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ELEMENTS

OF THE

DIFFERENTIAL AND INTEGRAL CALCULUS

WITH APPLICATIONS

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ELEMENTS

OF THE

DIFFERENTIAL AND INTEGRAL CALCULUS

WITH APPLICATIONS

BY

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PREFACE TO SECOND EDITION, REVISED.

In this edition, Chapters I, IV and V have been entirely rewritten.

An effort has been made to present the introduction to the subject as clearly and simply as possible, in order that the student may be given at the very beginning of the subject some real grasp of its essential ideas.

Somewhat more attention has also been given to infinitesimals.

A large number of new problems have been added. The answers to some of the problems are given and the answers to other problems have been purposely omitted. It is desirable often that the student may have the satisfaction of knowing that he is probably right when he secures the answer that is given. And it is frequently preferable to omit the answers, particularly in integration problems where the answers might suggest the methods to be employed.

WM. S. HALL.

Lafayette College, Easton, Pa.,
Nov., 1921.

PREFACE TO FIRST EDITION.

THIS work is an introduction to the study of the Differential and Integral Calculus, and is intended for colleges and technical schools. The object has been to present the Calculus and some of its important applications simply and concisely, and yet to give as much as it is necessary to know in order to enter upon the study of those subjects which presume a knowledge of the Calculus. The book will be found to be adapted to the needs of the mathematical student, and also will enable the engineer to get that knowledge of the Calculus which is required by him in order to make practical applications of the subject.

All of the formulas for differentiation are established by the method of limits. This method is preferred because it is more readily understood, and is more rigorous than the method of infinitesimals; and, moreover, it has the great advantage of being a familiar method, as the student has previously used it in Algebra and Geometry. But the differential notation is fully explained, and is employed when there is any advantage gained by so doing, particularly in the investigations of the Integral Calculus.

As soon as the fundamental formulas of differentiation have been established, the corresponding inverse operations or integrations follow. Thus the essential unity of the two branches of the Calculus is emphasized, the whole subject is made more intelligible, and there is a saving of much space.

Principal applications of the Calculus, as in Maxima and Minima,

Radius of Curvature, etc., are treated at some length, while less important subjects are treated much more briefly.

A large number of carefully selected examples, some original ones, and numerous practical numerical problems from mechanics and different branches of applied mathematics are given.

As there has been an increasing demand for a short course in Differential Equations, a chapter on this subject is given which it is hoped will meet a much-felt want.

A table of Integrals, arranged for convenience of reference, is appended.

Many American and English books, and some of the leading French and German works, have been freely consulted, and problems have been gathered from many different sources.

WILLIAM S. HALL.

EASTON, PA., January, 1897.

CONTENTS.

CHAPTER I.

Definitions and First Principles.

Art.		Page
1.	Constants and Variables	1
2.	Functions	1
3.	Increments and Continuous Functions	3
4.	Limits	4
5.	Limiting Ratio of Increments	7
6.	Derivatives	10
7.	Differentiation and the Differential Calculus	12

CHAPTER II.

Differentiation of Algebraic Functions.

8.	Definition	15
9.	Algebraic Sum of Functions	15
10.	Product of a Constant and a Function	16
11.	Any Constant	16
12.	Product of Two Functions	17
13.	Product of Three or More Functions	17
14.	Quotient of Two Functions	18
15.	Constant Power of a Function. Problems	19

CHAPTER III.

Differentiation of Transcendental Functions.

16.	Definitions	23
17.	Base of the Natural System of Logarithms	24
18.	Logarithmic Functions	24

ART.		PAGE
19.	Exponential Function with Constant Base	25
20.	Exponential Function with Variable Base. Problems	26
21.	Circular Measure	28
22.	Limiting Value of $\frac{\sin\theta}{\theta}$	29
23.	Trigonometric Functions	30
24.	Inverse Trigonometric Functions. Problems	33

CHAPTER IV.

DIFFERENTIALS AND RATES.

25.	Definition	38
26.	Geometric Derivation of Formulas. Problems	40

CHAPTER V.

INTEGRATION.

27.	Definition	45
28.	Fundamental Formulas	46
29.	Elementary Rules	47
30.	Constant of Integration	48
31.	Integration of Trigonometric Differentials. Problems	53
32.	Definite Integrals	54-a
33.	Geometric Illustration of Definite Integration	54-c
34.	Change of Limits. Problems	54-e
35.	Variable of Integration	54-g
36.	Miscellaneous Problems	54-g

CHAPTER VI.

SUCCESSIVE DIFFERENTIATION AND INTEGRATION.

37.	Successive Derivatives	55
38.	Successive Integration. Problems	56

APPLICATIONS IN MECHANICS.

39.	Velocity and Acceleration of Motion	59
40.	Uniformly Accelerated Motion. Problems	60
41.	Derivatives of the Product of Two Functions. Problems	61

CHAPTER VII.

Functions of Two or More Variables. Implicit Functions. Change of the Independent Variable.

Art.		Page
42.	Partial Differentiation	64
43.	Total Differential of a Function of Two or More Independent Variables	65
44.	Total Derivative when $u=f(x, y, z)$, $y=\phi(x)$, and $z=\phi_1(x)$	66
45.	Successive Partial Derivatives	68
46.	If $u=f(x, y)$, to prove that $\frac{\partial^2 u}{\partial y \partial x}=\frac{\partial^2 u}{\partial x \partial y}$	69
47.	Implicit Functions. Problems	69
48.	Integration of Functions of Two or More Variables	71
49.	Integration of Total Differentials of the First Order. Problems	72
50.	Change of the Independent Variable. Problems	73

CHAPTER VIII.

Development of Functions.

51.	Definition	78
52.	Maclaurin's Theorem. Problems	78
53.	Taylor's Theorem	83
54.	Demonstration of Taylor's Theorem. Problems	84
55.	Rigorous Proof of Taylor's Theorem	87
56.	Remainder in Taylor's and Maclaurin's Theorems	88
57.	Taylor's Theorem for Functions of Two or More Independent Variables	89

CHAPTER IX.

Evaluation of Indeterminate Forms.

58.	Indeterminate Forms	91
59.	Functions that take the Form $\frac{0}{0}$. Problems	92
60.	Functions that take the Form $\frac{\infty}{\infty}$	94
61.	Functions that take the Forms $0 \times \infty$ and $\infty - \infty$	95
62.	Functions that take the Forms 0^0, ∞^0 and $1^{\pm\infty}$. Problems	96
63.	Compound Indeterminate Forms. Problems	98

CHAPTER X.

Maxima and Minima of Functions.

64.	Definitions and Geometric Illustration	99
65.	Method of Determining Maxima and Minima	100

ART. PAGE
66. Conditions for Maxima and Minima by Taylor's Theorem. Problems . 101
67. Maxima and Minima of Functions of Two Independent Variables . . 111
68. Maxima and Minima of Functions of Three Independent Variables. Problems 113

CHAPTER XI.

TANGENTS, NORMALS AND ASYMPTOTES.

69. Equations of the Tangent and Normal 116
70. Lengths of Tangent, Normal, Subtangent and Subnormal . . . 117
71. Tangent of the Angle between the Radius Vector and the Tangent to a Plane Curve in Polar Coördinates 117
72. Derivative of an Arc 118
73. Derivative of an Arc in Polar Coördinates 119
74. Lengths of Tangent, Normal, etc., in Polar Coördinates. Problems . 120
75. Rectilinear Coördinates 122
76. Asymptotes parallel to the Axis 123
77. Asymptotes determined by Expansion 124
78. Asymptotes in Polar Coördinates. Problems 124

CHAPTER XII.

DIRECTION OF CURVATURE. POINTS OF INFLECTION. RADIUS OF CURVATURE. CONTACT.

79. Direction of Curvature 126
80. Direction of Curvature in Polar Coördinates 128
81. Points of Inflection. Problems 129
82. Curvature. Problems 130
83. Radius of Curvature 131
84. Radius of Curvature in Polar Coördinates 132
85. Contact of Different Orders 133
86. Radius of Osculating Circle and Coördinates of Centre 134
87. Osculating Circle and Contact of the Third Order. Problems . . 135

CHAPTER XIII.

EVOLUTES AND INVOLUTES. ENVELOPES.

88. Definition 138
89. Equation of the Evolute 138
90. A Normal to Any Involute is Tangent to its Evolute 140

ART.		PAGE
91.	The Difference between any Two Radii of Curvature of an Involute	141
92.	Mechanical Construction of an Involute	142
93.	Envelopes of Curves	142
94.	Equation of the Envelope of a Family of Curves. Problems	143

CHAPTER XIV.

SINGULAR POINTS.

95.	Definitions	147
96.	Multiple Points. Problems	147
97.	Cusps	150
98.	Conjugate Points. Stop Points. Shooting Points	151

CHAPTER XV.

INTEGRATION OF RATIONAL FRACTIONS.

99.	Rational Fractions. Problems	154

CHAPTER XVI.

INTEGRATION OF IRRATIONAL FUNCTIONS.

100.	Irrational Functions	160
101.	Irrational Functions containing only Monomial Surds. Problems	160
102.	Functions containing only Binomial Surds of the First Degree. Problems	161
103.	Functions having the Form $\dfrac{x^{2m+1}dx}{(a+bx^2)^{\frac{m}{2}}}$	162
104.	Functions having the Form $f\left(x, \sqrt[n]{\dfrac{ax+b}{cx+d}}\right)dx$. Problems	163
105.	Functions containing only Trinomial Surds of the Form $\sqrt{a+bx+cx^2}$. Problems	164
106.	Binomial Differentials	166
107.	Conditions of Integrability of Binomial Differentials. Problems	166

CHAPTER XVII.

INTEGRATION BY PARTS AND BY SUCCESSIVE REDUCTION.

108.	Integration by Parts. Problems	170
109.	Formulas of Reduction. Problems	171

CHAPTER XVIII.

Integration of Transcendental Functions. Integration by Series.

Art.		Page
110.	Introduction	177
111.	Integration of the Form $\int f(x)(\log x)^n dx$. Problems	177
112.	Integration of the Form $\int x^m a^{nx} dx$. Problems	178
113.	Integration of the Form $\int \sin^m\theta \cos^n\theta\, d\theta$. Problems	179
114.	Integration of the Forms $\int x^n \cos(ax)dx$ and $\int x^n \sin(ax)dx$ . . .	182
115.	Integration of the Forms $\int e^{ax} \sin^n x\, dx$ and $\int e^{ax} \cos^n x\, dx$. Problems .	182
116.	Integration of the Forms $\int f(x)$ arc sin $x\, dx$, $\int f(x)$ arc cos $x\, dx$, etc. Problems	184
117.	Integration of the Form $\int \frac{d\theta}{a + b\cos\theta}$. Problems	184
118.	Integration by Series. Problems	186

CHAPTER XIX.

Integration as a Summation. Areas and Lengths of Plane Curves.

119.	Integration as a Summation. Problems	187
120.	Areas of Plane Curves in Polar Coördinates. Problems . . .	191
121.	Rectification of Plane Curves referred to Rectangular Axes. Problems	192
122.	Rectification of Curves in Polar Coördinates. Problems . . .	195
123.	The Common Catenary	196

CHAPTER XX.

Surfaces and Volumes of Solids.

124.	Surfaces and Volumes of Solids of Revolution. Problems . . .	199
125.	Surfaces by Double Integration	202
126.	Volumes by Triple Integration. Problems	203

CHAPTER XXI.

Centre of Mass. Moment of Inertia. Properties of Guldin.

127.	Definitions	206
128.	General Formulas for Centre of Mass. Problems	207
129.	Centre of Mass for Plane Surfaces. Problems	210

ART.		PAGE
130.	Centre of Mass of Surfaces of Revolution. Problems	212
131.	Centre of Mass of Solids of Revolution. Problems	213
132.	Moments of Inertia of Surfaces. Problems	214
133.	Guldin's Theorems. Problems	215

CHAPTER XXII.

DIFFERENTIAL EQUATIONS.

134.	Definition	217
135.	Differential Equations of the First Order and Degree. Problems	218
136.	Homogeneous Differential Equations. Problems	219
137.	The Form $(ax + by + c)dx + (a'x + b_1y + c')dy = 0$. Problems	221
138.	The Linear Equation of the First Order. Problems	222
139.	Extension of the Linear Equation. Problems	224
140.	Exact Differential Equations. Problems	225
141.	Factors Necessary to make Differential Equations Exact. Problems	227
142.	First Order and Degree with Three Variables. Problems	230
143.	First Order and Second Degree. Problems	232
144.	Differential Equations of the Second Order. Problems	234

APPENDIX.

Table of Integrals	238
Note A	250

DIFFERENTIAL AND INTEGRAL CALCULUS.

CHAPTER I.

DEFINITIONS AND FIRST PRINCIPLES.

ART. 1. CONSTANTS AND VARIABLES.

The quantities employed in the Calculus belong to two classes,—*constants* and *variables*.

A *constant quantity* is one that retains the same value throughout a discussion. Constants are *numerical* or *literal*. Literal constants have values that are assigned arbitrarily and are usually denoted by the first letters of the alphabet, as a, b, c, etc.

A *variable quantity* is one that admits of an infinite number of values in the same discussion within limits determined by the nature of the problem. Variables are usually represented by the last letters of the alphabet, as u, v, w, x, y and z.

ART. 2. FUNCTIONS.

One variable quantity is a function of another when they are so related that for any assigned value of the latter there is a corresponding value of the former. Arbitrary values may be assigned to the second variable, which is then called the *independent variable*, while the first variable is called the *dependent variable* or *function*.

Only functions that may be given definite mathematical expression are treated here.

For example, the area of a circle is a function of its diameter because the area depends on the length of the diameter, and the diameter whose length may be assigned at pleasure is the independent variable.

The price of any article is a function of the cost of its production; the speed of a body falling from rest is a function of the time that it has fallen.

The trigonometric functions are functions of the angle, the angle being regarded as the independent variable.

Expressions involving x, such as

$$x^3,\ ax^2 + bx + c,\ \log x,\ \sqrt{1 - x^2},$$

are functions of the independent variable x.

A quantity may be a function of two or more variables. For example, the area of a plane triangle is a function of its base and altitude; the volume of a rectangular parallelopiped is a function of its three dimensions.

The expressions,

$$x^3 - 3x^2y^2 + y^3,\ \sqrt{a^2x^2 + b^2y^2},\ a^{x+y},$$

are functions of x and y.

The expressions,

$$a^2x^2 + b^2y^2 + c^2z^2,\ \log(x^2 + xy + z^2),$$

are functions of x, y and z.

An *explicit function* is one whose value is expressed directly in terms of the independent variable and constants. For example, y is an explicit function of x in the equations

$$y = \frac{b}{a}\sqrt{x^2 - a^2}, \text{ and } y = 2\,ax + x^2 + c^2.$$

Explicit functions are denoted by such symbols as the following:

$$y = f(x);\ y = F(x);\ y = \phi(x);\ y = f'(x);$$

which may be read respectively: "y equals the f function of x"; "y equals the large F function of x"; "y equals the ϕ function of x"; "y equals the f prime function of x."

When the equation giving the relation connecting the variables is not solved with reference to y, y is an *implicit function* of x. For example, y is an implicit function of x in the equations

$$a^2y^2 + b^2x^2 = a^2b^2,\ ax^2 + bxy + cy^2 = 0.$$

Implicit functions are denoted by such symbols as the following:

$$f(x, y) = 0;\ F(x, y) = 0;\ \phi(x, y) = 0;$$

which may be read, "the f function of x and y equals zero"; etc.

If

$$f(x) = x^3 + 2x^2 + x + 1,$$

when the independent variable $x = 3$,

$$f(3) = 27 + 18 + 3 + 1 = 49.$$

If

$$\phi(x, y) = \sin x \cos y + \cos x \sin y,$$

when $x = a$, and $y = b$,

$$\phi(a, b) = \sin a \cos b + \cos a \sin b.$$

ART. 3. INCREMENTS AND CONTINUOUS FUNCTIONS.

The amount by which a variable is changed when it passes from one value to another is called an *increment*, and is found by subtracting the first value from the second. Thus an increment received by the variable x would be denoted by Δx, and would be read "delta x," or "increment of x." The increment of a variable may be either positive or negative; if it is positive the variable is increasing, and if it is negative the variable is decreasing. A negative increment is sometimes called a *decrement*.

In the expression, $y = f(x)$, x is commonly taken as the independent variable, and if it takes an increment Δx, the corresponding increment of y is denoted by Δy, giving

$$y + \Delta y = f(x + \Delta x).$$

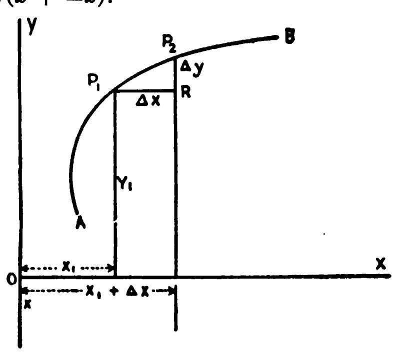

FIG. 1.

In Fig. 1, let the curve AB be the locus of the equation $y = f(x)$, and the coördinates of P_1 and P_2 be (x_1, y_1) and $(x_1 + \Delta x,\ y_1 + \Delta y)$ respectively.

When x_1 receives an increment Δx, it will be seen that y_1 receives a corresponding increment Δy.

A variable is said to be a *continuous variable* through the interval between two of its values if in passing from one to the other it assumes all intermediate values in the order of their magnitude. And a *continuous function* is one whose increment approaches zero as the in-

crement of the independent variable approaches zero. In Fig. 1, y is a continuous function from P_1 to P_2 for Δy approaches zero as Δx approaches zero, causing the point P_2 to approach P_1.

In the graph of the function $y = \frac{1}{x}$, (Fig. 3), when negative values are assigned to x, a branch of the curve is constructed in the third quadrant, and when positive values are assigned to x a branch of the curve is constructed in the first quadrant. As the negative values of x increase toward zero the branch of the curve in the third quadrant approaches the lower part of the y-axis but never reaches it, and as x approaches zero from the right the branch of the curve in the first quadrant approaches the upper part of the y-axis but never reaches it. Thus it will be seen that the curve is *discontinuous* for the value $x = 0$, but continuous for all other values of x.

Art. 4. Limits.

A *limit of a variable* is a constant value which the variable continually approaches, and from which it can be made to differ by a quantity less than any assignable quantity, but which it cannot absolutely equal.

For example, assume that a body is moving along a straight line from A to B as in Fig. 2, under the condition that in the first interval

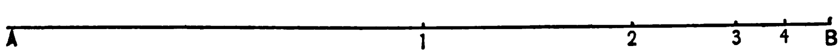

Fig. 2.

of time it shall move one-half of the entire distance, or from A to 1, and one-half of the remaining distance, or from 1 to 2, in the second interval, and so on, moving during each interval one-half of the distance remaining. In this case the entire distance AB is a constant toward which the distance traversed by the moving point continually approaches as a limit but which it never reaches.

The limit of a variable which becomes less than any assignable quantity is *zero*.

The limit of a variable which becomes greater than any assignable quantity is *infinity*.

The limit of $\frac{1}{x}$, as x increases indefinitely, is zero; as x in this fraction increases the fraction decreases, and as x may be increased at pleasure, the fraction may be made to approach indefinitely near to zero.

Let the locus of the equation $y = \frac{1}{x}$ be drawn by the method of rectangular coördinates as in Fig. 3.

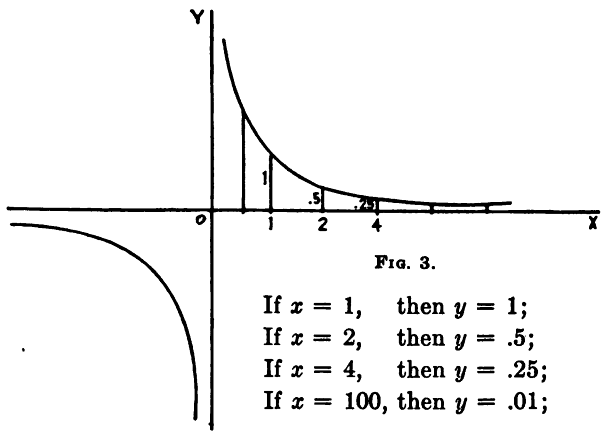

FIG. 3.

If $x = 1$, then $y = 1$;
If $x = 2$, then $y = .5$;
If $x = 4$, then $y = .25$;
If $x = 100$, then $y = .01$;
... ...

Or as the abscissa increases the ordinate decreases toward zero as a limit; thus the curve continually approaches the x-axis, but never reaches it.

The limit of the value of the repeating decimal 0.555..., as the number of decimal places is continually increased, is $\frac{5}{9}$.

A variable may approach its limit in three ways:

1st. A variable may increase toward its limit, as is the case when a polygon is inscribed in a circle; the polygon will increase toward the circle as its limit, as the number of sides is increased.

2d. A variable may decrease toward its limit, as is the case when a polygon is circumscribed about a circle; the polygon will decrease toward the circle as its limit, as the number of sides is increased.

3d. A variable may approach its limit and be sometimes greater and sometimes less than its limit. For example, take the geometrical progression whose first term is 1 and whose ratio is $-\frac{1}{3}$, giving the

series $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots$; here the limit of the sum of the series, as the number of terms is indefinitely increased, is $\frac{3}{4}$; but the sum of any odd number of terms will be greater than this limit, and the sum of any even number of terms will be less.

As the independent variable approaches a limit, the function or dependent variable may approach a limit. Thus in the equation,

$$y = \frac{2}{x} + x,$$

if x approaches 1 as a limit, y approaches 3 as a limit.

In the equation,

$$y = \frac{\cos\theta}{1 + \sin\theta},$$

if θ approaches 0 as a limit, y approaches 1 as a limit.

$\theta \doteq 0$ is a convenient notation for the expression "θ approaches 0 as a limit."

From the definition of a limit of a variable, it follows that the difference between the variable and its limit is a variable which has zero for its limit. Therefore, to prove that a given constant is the limit of a certain variable, it is sufficient to show that the difference between the variable and the constant has the limit zero.

1st. The proposition in the theory of limits used most frequently in the Calculus is the following:

If two variables are equal and are so related that as they change they remain always equal to each other, and each approaches a limit, their limits are equal.

Let x and y be the variables, and a and b their respective limits, and let x' and y' represent the differences between the variables and their limits.

Then $$a = x + x', \text{ and } b = y + y'.$$

Since $x = y$ is always true,

$$a - b = x' - y'. \tag{1}$$

In equation (1), $x' - y'$ is equal to a constant, and x' and y' are variables that approach zero as a limit. Hence $x' - y' = 0$, and, therefore, $a - b = 0$, or $a = b$.

The supplementary propositions are readily established.

2d. The limit of the algebraic sum of a finite number of variables is the algebraic sum of their limits.

3d. The limit of the product of two or more variables is the product of their limits.

4th. The limit of the quotient of two variables is the quotient of their limits.

5th. The limit of the product of a constant and a variable is the product of the constant and the limit of the variable.

6th. The limit of a power of a variable is that power of its limit.

The following special limiting values occur frequently:

If $n > 0$ and $a > 0$,

(1) $\lim\limits_{x \doteq \infty}\left[nx\right] = \infty$:

(2) $\lim\limits_{x \doteq 0}\left[\frac{n}{x}\right] = \infty$;

(3) $\lim\limits_{x \doteq \infty}\left[\frac{n}{x}\right] = 0$;

(4) $\lim\limits_{x \doteq \infty}\left[\frac{x}{n}\right] = \infty$;

(5) $\lim\limits_{x \doteq \infty}\left[a^x\right] = 0$ if $a < 1$, and $= \infty$ if $a > 1$;

(6) $\lim\limits_{x \doteq -\infty}\left[a^x\right] = \infty$ if $a < 1$, and $= 0$ if $a > 1$;

(7) $\lim\limits_{x \doteq 0}\left[\log_a x\right] = \infty$ if $a < 1$, and $= -\infty$ if $a > 1$;

(8) $\lim\limits_{x \doteq 0}\left[\log_a x\right] = -\infty$ if $a < 1$, and $= \infty$ if $a > 1$.

Art. 5. Limiting Ratio of Increments.

If an increment be given to x in $y = f(x)$, y will receive a corresponding increment; required the limiting value of the ratio $\frac{\Delta y}{\Delta x}$.

Taking first a particular function, for example, $y = ax^2$.

In this example, if x receives an increment represented by Δx or h, y will take a corresponding increment represented by Δy; and the equation becomes

$$y + \Delta y = a(x + h)^2$$
$$= ax^2 + 2axh + ah^2.$$

Subtracting $$y = ax^2,$$

$$\Delta y = 2axh + ah^2, \tag{1}$$

Dividing by $$\Delta x = h,$$

$$\frac{\Delta y}{\Delta x} = 2ax + ah. \tag{2}$$

As h approaches zero, each member of this equation will approach a limit, and by Art. 4 these limits are equal; therefore

$$\text{limit of } \frac{\Delta y}{\Delta x} = 2ax. \tag{3}$$

In order to make a definite application, let $a = 16$ in the given equation, and substitute s for y, and t for x; then the equation becomes $s = 16t^2$, which is approximately the equation of a freely falling body near the earth's surface, s representing the number of feet fallen, and t the time of the fall in seconds.

Then the proper substitutions made in equation (3) give

$$\text{limit of } \frac{\Delta s}{\Delta t} = 32t,$$

which is seen to be the actual velocity at the end of t seconds. Therefore the limiting ratio of the increments of distance and time is the velocity at the end of the period.

To illustrate further, let the object be to determine the increments produced in s by certain decreasing increments assigned to t, when t has some given value, as 10.

Substituting $s = y$, $t = x = 10$, $a = 16$ and $\Delta t = \Delta x$, in (1), (2) and (3):

$$\Delta s = 320\,\Delta t + 16(\Delta t)^2,$$

$$\frac{\Delta s}{\Delta t} = 320 + 16(\Delta t),$$

and $$\text{limit of } \frac{\Delta s}{\Delta t} = 320.$$

Let $\Delta t = 0.1$, then $\Delta s = 32.16$ and $\frac{\Delta s}{\Delta t} = 321.6$;

Let $\Delta t = 0.01$, then $\Delta s = 3.2016$ and $\frac{\Delta s}{\Delta t} = 320.16$;

Let $\Delta t = 0.001$, then $\Delta s = .320016$ and $\frac{\Delta s}{\Delta t} = 320.016$;

Let $\Delta t = 0.0001$, then $\Delta s = .03200016$ and $\frac{\Delta s}{\Delta t} = 320.0016$:

... ...

And it is apparent that as Δt continually diminishes, Δs also decreases, and the ratio of the increments, $\frac{\Delta s}{\Delta t}$, approaches 320 as its limit.

Take next a geometrical example.

Let the curve be the parabola whose equation is $y = \sqrt{2x}$, and whose locus is shown in Fig. 4. Let (x', y') be the coördinates of P, and $(x' + \Delta x, y' + \Delta y)$ be the coördinates of any second point P'.

Then $$y' + \Delta y = \sqrt{2(x' + \Delta x)}. \qquad (1)$$

Subtracting $$y' = \sqrt{2x'},$$
$$\Delta y = \sqrt{2(x' + \Delta x)} - \sqrt{2x'};$$

hence $$\frac{\Delta y}{\Delta x} = \frac{\sqrt{2(x' + \Delta x)} - \sqrt{2x'}}{\Delta x}. \qquad (2)$$

Rationalizing the numerator of (2),

$$\frac{\Delta y}{\Delta x} = \frac{2\Delta x}{\Delta x[\sqrt{2(x' + \Delta x)} + \sqrt{2x'}]} = \frac{2}{\sqrt{2(x' + \Delta x)} + \sqrt{2x'}},$$

and $$\text{limit of } \frac{\Delta y}{\Delta x} = \frac{2}{2\sqrt{2x'}} = \frac{1}{\sqrt{2x'}} = \frac{1}{y'}.$$

From the figure, it is obvious that $\frac{\Delta y}{\Delta x}$ is the tangent of the angle $P'TN$, and if the point P' approaches indefinitely near to P, the line $P'T$ will be a tangent to the curve at P. Therefore, the limit of $\frac{\Delta y}{\Delta x}$, as

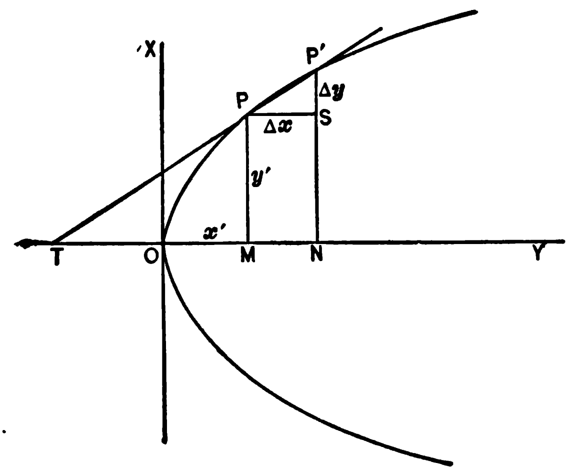

FIG. 4.

Δx approaches zero, is the tangent of the angle which the curve makes with the X-axis, and is equal to the reciprocal of the ordinate of the point of contact.

If P is at the extremity of the latus rectum, coördinates $(\frac{1}{2}, 1)$, then

$$\text{limit of } \frac{\Delta y}{\Delta x} = 1 = \tan 45^\circ,$$

or the tangent to the parabola at the extremity of the latus rectum makes an angle of 45° with the X-axis, which is a well-known property of the curve.

ART. 6. DERIVATIVES.

The limit of $\frac{\Delta y}{\Delta x}$ in the preceding article is called the *derivative* of y with respect to x, and is denoted by $\frac{dy}{dx}$. Hence the definition: *If y is a function of x, the derivative of y with respect to x is the limiting value of the ratio of the increment of y to the corresponding increment of x, as the increment of x approaches zero.*

If $y = f(x)$ is the equation of any plane curve, such as MN in Fig. 5, the derivative of the function may be shown geometrically as follows:

Let the rectangular coördinates of the point R of the curve MN be (x, y) and the coördinates of another point S be $(x + \Delta x, y + \Delta y)$.

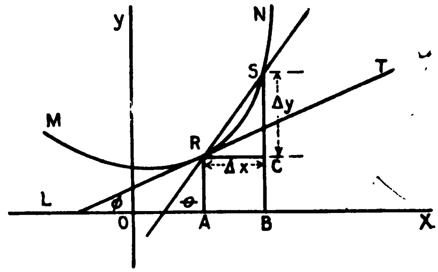

FIG. 5.

1. *Addition of increments,*

$$y + \Delta y = f(x + \Delta x) = BS.$$

2. *Solving for Δy,*

$$\Delta y = f(x + \Delta x) - f(x) = CS.$$

3. *Ratio of increments,*

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{CS}{RC}$$

$$= \tan \theta = \text{slope of secant } RS.$$

4. *Limiting ratio,*

$$\lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right] = \lim_{\Delta x \doteq 0}\left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right],$$

$$\text{or } \frac{dy}{dx} = \text{derivative of the function.}$$

As $\Delta x \doteq 0$, Δy also approaches zero, and the point S approaches the point R, the secant revolves about the point R and approaches the tangent line as its limiting position, and

$$\lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right] = \lim_{\Delta x \doteq 0}\left[\tan \theta\right] = \tan \phi,$$

$$\text{or, } \frac{dy}{dx} = \tan \phi$$

$$= \text{slope of tangent at } R.$$

The ratio $\frac{\Delta y}{\Delta x}$ may be regarded as the average *rate of change* of y with respect to x in the interval Δx, and $\frac{dy}{dx}$ may be considered as the *rate of change of the function at a point.*

Hence, the derivative of the ordinate y with respect to the abscissa x at any point of a curve is the slope of the tangent drawn to the curve at that point.

The *derivative* is sometimes called the *differential coefficient*, and the symbols $f'(x)$ and $D_x y$ are frequently used instead of $\frac{dy}{dx}$.

The term "derivative" is fully as significant as differential coefficient, and is certainly to be preferred when the method of limits is used. The word "derivative" will generally be used in this book. *Derived function* is another name which is sometimes adopted instead of the word "derivative." It must be carefully noticed that Δ and d are not factors, but symbols of operations.

ART. 7. DIFFERENTIATION AND THE DIFFERENTIAL CALCULUS

The operation of finding the derivative of a function is called *differentiation.* All the functions considered in this book are differentiable.

The *differential calculus* is that branch of mathematics in which the properties and relations of functions are investigated and applied by aid of derivatives.

The method of differentiation is given in the following general rule:

FIRST. *Substitute $x + \Delta x$ for x, giving the corresponding function a value $y + \Delta y$;*

SECOND. *Obtain the increment of the function, Δy, by subtracting its first value from the second;*

THIRD. *Divide the increment of the function by the increment of the independent variable, giving $\frac{\Delta y}{\Delta x}$;*

FOURTH. *Find the limit of this ratio as $\Delta x \doteq 0$, giving $\frac{dy}{dx}$.*

PROBLEMS.

1. In the equation $y = x^2 - 3x + 5$, what is the increment received by y if an increment of 1 is given to x, when $x = 3$? *Ans.* 4.

2. In the equation $y = mx + n$, what is the ratio of the increment of the ordinate to the increment of the abscissa? *Ans.* m.

3. In the equation $y^2 = \frac{4}{9}(18x - x^2)$, what are the increments received by y corresponding to an increment of 1 given to the abscissa, starting from $x = 3$? *Ans.* $-2\sqrt{5} \pm \frac{4}{3}\sqrt{14}$.

4. If $f(x) = x^4 - 2x^2 + 3x - 1$, find $f(0)$, $f(1)$ and $f(-2)$.

5. If $F(x) = x(x-2)(x-3)$, find $F(2)$, $F(\frac{1}{2})$ and $F(5)$.

6. Given $\phi(x) = \tan x$; prove that $\phi(2x) = \dfrac{2\phi(x)}{1 - [\phi(x)]^2}$.

7. Prove the following limits:

(a) $\lim\limits_{y \doteq 1}\left[\dfrac{y^2 - 2y + 5}{y^2 + 7}\right] = \frac{1}{2}$.

(b) $\lim\limits_{x \doteq \infty}\left[\dfrac{x^2 + x}{4x^2 + 3x - 2}\right] = \frac{1}{4}$.

(c) $\lim\limits_{x \doteq \infty}\left[\dfrac{x(x+2)}{(x+3)(x+4)}\right] \doteq 1$.

(d) $\lim\limits_{\Delta x \doteq 0}\left[\dfrac{(x + \Delta x)^n - x^n}{\Delta x}\right] = nx^{n-1}$.

(e) $\lim\limits_{x \doteq 0}\left[\dfrac{\sin 2x}{\tan x}\right] = 2$.

8. If $y = 4x^2 + 3x - 2$, find the value of Δy corresponding to Δx.

9. Given $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, express y as an explicit function of x.

10. If $y = f(x) = x + \dfrac{1}{x}$, find $f(x + 1)$.

11. Find the slope of the parabola $y = x^2 - 2x$, at the point $(4, 8)$.

12. Obtain the derivatives of the following functions:

(a) $y = x^3$;

(b) $y = x^2 + x$;

(c) $y = (x + 1)(x + 2)$.

13. If y is the area of a circle whose radius is x, show that $\dfrac{dy}{dx}$ is equal to the circumference.

The Differential and Integral Calculus originated in the seventeenth century. Newton was the first discoverer of the new analysis, but to Leibnitz belongs the credit of priority of publication and the invention of a notation much superior to Newton's, and which has entirely superseded it. Leibnitz first published his new method in 1684.

Newton called his method the method of fluxions. According to him, all

quantities are supposed to be generated by continuous motion, as a line by a moving point. Fluxions are the relative rates with which functions and the variables on which they depend are increasing at any instant.

Leibnitz considered all quantities to be made up of indefinitely small parts or infinitesimals; a surface being composed of indefinitely small parallelograms, and a volume of indefinitely small parallelopipeds. The nomenclature and notation of the Calculus now in common use were original with Leibnitz, and introduced by him.

According to Newton, the fluxion of x would be denoted by $\dot{x}$, while by Leibnitz the corresponding derivative is $\frac{dx}{dt}$.

CHAPTER II.

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS.

ART. 8. DEFINITION.

An *algebraic function* is one in which the only indicated operations are addition, subtraction, multiplication, division, and involution and evolution with constant exponents.*

In this chapter only functions of a single independent variable x will be treated, and throughout the chapter u, v and w will be regarded as functions of x.

ART. 9. ALGEBRAIC SUM OF ANY NUMBER OF FUNCTIONS.

If y be taken to represent the algebraic sum of three functions of x, the equation may be written

$$y = u + v - w. \tag{1}$$

If an increment Δx is given to x, the variables y, u, v and w, which are functions of x, will take the corresponding increments Δy, Δu, Δv and Δw, respectively; then (1) becomes

$$y + \Delta y = (u + \Delta u) + (v + \Delta v) - (w + \Delta w). \tag{2}$$

Subtracting (1) from (2),

$$\Delta y = \Delta u + \Delta v - \Delta w. \tag{3}$$

Dividing by Δx,
$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} - \frac{\Delta w}{\Delta x}. \tag{4}$$

When Δx approaches zero,

$$\text{limit of } \frac{\Delta y}{\Delta x} = \frac{dy}{dx}, \text{ limit of } \frac{\Delta u}{\Delta x} = \frac{du}{dx}, \text{ etc., by Art. 6.}$$

* In this definition of an algebraic function it is understood that the operations are not repeated an infinite number of times.

Therefore $$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}, \text{ by Art. 5, 1st and 2d;}$$

or $$\frac{d(u + v - w)}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}. \qquad \text{I.}$$

If the algebraic sum of four or more variables be given, the derivative would be found similarly.

I. Hence, *the derivative of the algebraic sum of any number of functions of x is equal to the algebraic sum of their derivatives.*

Art. 10. Product of a Constant and a Function.

Let a represent any constant, then the product of a constant and a function of x may be written

$$y = av. \qquad (1)$$

Let v and y take the increments Δv and Δy corresponding to the increment Δx given to x, then

$$y + \Delta y = a(v + \Delta v). \qquad (2)$$

Subtracting (1) from (2), $\Delta y = a\Delta v$.

Dividing by Δx, $$\frac{\Delta y}{\Delta x} = a\frac{\Delta v}{\Delta x}. \qquad (3)$$

When Δx approaches zero, by Art. 5, 1st,

$$\text{limit } \frac{\Delta y}{\Delta x} = \text{limit}\left(a\frac{\Delta v}{\Delta x}\right);$$

therefore $$\frac{dy}{dx} = a\frac{dv}{dx}, \text{ by Art. 6.} \qquad \text{II.}$$

If $v = x$, $\Delta v = \Delta x$ and $\dfrac{dy}{dx} = a$.

II. Hence, *the derivative of the product of a constant and a function of x is the product of the constant and the derivative of the variable.*

Art. 11. Any Constant.

As the value of a constant remains unchanged in any one discussion, the constant receives no increment, or, in other words, the increment of the constant is zero.

Let a represent any constant; then

$$\Delta a = 0 \quad \text{and} \quad \frac{\Delta a}{\Delta x} = 0.$$

Therefore, when Δx approaches zero,

$$\frac{da}{dx} = 0. \qquad \text{III.}$$

III. Hence, *the derivative of a constant is zero.*

ART. 12. PRODUCT OF TWO FUNCTIONS.

Let the product of two functions of x be represented by $y = uv$. When x is given an increment, the variables v, u and y receive corresponding increments, and the equation becomes

$$y + \Delta y = (u + \Delta u)(v + \Delta v)$$
$$= uv + u\Delta v + v\Delta u + \Delta u\,\Delta v. \qquad (1)$$

Hence $$\Delta y = (v + \Delta v)\,\Delta u + u\Delta v; \qquad (2)$$

and $$\frac{\Delta y}{\Delta x} = (v + \Delta v)\frac{\Delta u}{\Delta x} + u\frac{\Delta v}{\Delta x}. \qquad (3)$$

When Δx approaches zero,

$$\text{limit}\,\frac{\Delta y}{\Delta x} = \frac{dy}{dx}, \qquad \text{limit}\,u\frac{\Delta v}{\Delta x} = u\frac{dv}{dx},$$

$$\text{limit}\,(v + \Delta v) = v, \quad \text{and} \quad \text{limit}\,\frac{\Delta u}{\Delta x} = \frac{du}{dx}.$$

Therefore, by Art. 4.

$$\frac{dy}{dx} = \frac{d\,(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}. \qquad \text{IV.}$$

IV. Hence, *the derivative of the product of two functions of x is the sum of the products of each function by the derivative of the other.*

ART. 13. PRODUCT OF THREE OR MORE FUNCTIONS.

Let the product of three functions of x be represented by $y = uvw$.

The product of two of the functions, as uv, may be taken equal to z; then, by the preceding article,

$$\frac{d(uvw)}{dx} = \frac{d(zw)}{dx} = w\frac{dz}{dx} + z\frac{dw}{dx}$$

$$= w\left(v\frac{du}{dx} + u\frac{dv}{dx}\right) + uv\frac{dw}{dx}$$

$$= wv\frac{du}{dx} + wu\frac{dv}{dx} + uv\frac{dw}{dx}. \qquad \text{V.}$$

This process may be extended to the differentiation of the product of any number of functions.

V. Hence, *the derivative of the product of any number of functions of x is equal to the sum of the products of the derivative of each into the product of all the others.*

ART. 14. QUOTIENT OF TWO FUNCTIONS.

Let the quotient of two functions of x be represented by $y = \frac{u}{v}$.

Then $$vy = u;$$

and, by V., $$v\frac{dy}{dx} + y\frac{dv}{dx} = \frac{du}{dx}.$$

Therefore $$\frac{dy}{dx} = \frac{\frac{du}{dx} - y\frac{dv}{dx}}{v}$$

$$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}. \qquad \text{VI.}$$

VI. Hence, *the derivative of a fraction is equal to the denominator multiplied by the derivative of the numerator, minus the numerator multiplied by the derivative of the denominator, divided by the square of the denominator.*

COR. 1. If the numerator is constant, $\frac{du}{dx} = 0$, by III., and VI. becomes

$$\frac{dy}{dx} = -\frac{u\frac{dv}{dx}}{v^2}.$$

Hence, the derivative of a fraction with a constant numerator is negative and equal to the numerator multiplied by the derivative of the denominator, divided by the square of the denominator.

COR. 2. If the denominator is constant, $\frac{dv}{dx} = 0$, by III., and VI. becomes

$$\frac{dy}{dx} = \frac{v\frac{du}{dx}}{v^2} = \frac{\frac{du}{dx}}{v},$$

which is the same result that would be obtained by II.

ART. 15. CONSTANT POWER OF A FUNCTION.

CASE 1. When the exponent is a positive integer.

Let v be a function of x, and n its exponent; then

$$y = v^n,$$

$$y + \Delta y = (v + \Delta v)^n,$$

and

$$\Delta y = (v + \Delta v)^n - v^n.$$

Expanding $(v + \Delta v)^n$, by the Binomial Theorem, and dividing by Δx,

$$\frac{\Delta y}{\Delta x} = \left[nv^{n-1} + \frac{n(n-1)}{1 \cdot 2}v^{n-2}(\Delta v) \cdots + (\Delta v)^{n-1}\right]\frac{\Delta v}{\Delta x}.$$

When Δx approaches zero, Δv approaches zero also; hence

$$\frac{dy}{dx} = nv^{n-1}\frac{dv}{dx}. \qquad \text{VII.}$$

CASE 2. When the exponent is a positive fraction, $\frac{m}{n}$.

Let

$$y = v^{\frac{m}{n}},$$

then

$$y^n = v^m.$$

As m and n are positive integers, by Case 1,

$$ny^{n-1}\frac{dy}{dx} = mv^{m-1}\frac{dv}{dx};$$

therefore

$$\frac{dy}{dx} = \frac{m}{n}\frac{v^{m-1}}{y^{n-1}}\frac{dv}{dx} = \frac{m}{n}\frac{v^{m-1}}{v^{m-\frac{m}{n}}}\frac{dv}{dx}$$

$$= \frac{m}{n}v^{\frac{m}{n}-1}\frac{dv}{dx}. \qquad \text{VII.}$$

CASE 3. When the exponent is negative and either integral or fractional, as $-n$.

Let $$y = v^{-n},$$

then $$y = \frac{1}{v^n}.$$

Differentiating by Art. 15, Cor. 1,

$$\frac{dy}{dx} = -\frac{nv^{n-1}\frac{dv}{dx}}{v^{2n}} = -nv^{-n-1}\frac{dv}{dx}. \qquad \text{VII.}$$

VII. *The derivative of a constant power of a function of x is equal to the product of the exponent, the function with its exponent diminished by unity, and the derivative of the function.*

Radical expressions may be differentiated according to this rule, the quantities being first transformed into equivalent expressions with fractional exponents.

The radical of the second order is the one that occurs most frequently. It is differentiated as follows:

Let $$y = \sqrt{v} = v^{\frac{1}{2}}.$$

$$\frac{dy}{dx} = \frac{1}{2}v^{-\frac{1}{2}}\frac{dv}{dx} = \frac{\frac{dv}{dx}}{2\sqrt{v}}.$$

Hence, the derivative of the square root of a function of x is equal to the derivative of the function divided by twice the square root of the function.

PROBLEMS.

The formulas established in this chapter are sufficient for the differentiation of all algebraic functions of a single variable.

Differentiate the following functions:

1. $y = a + bx + x^3.$

$$\frac{dy}{dx} = \frac{d}{dx}(a + bx + x^3)$$

$$= \frac{da}{dx} + \frac{d\,(bx)}{dx} + \frac{d\,(x^3)}{dx}, \text{ by I.}$$

$\frac{da}{dx}=0$, by III.; $\frac{d(bx)}{dx}=b$, by II.; $\frac{d(x^3)}{dx}=3x^2$, by VII.

Therefore $\frac{dy}{dx}=b+3x^2$.

2. $y=(a+x)(b+2x^2)$.

$$\frac{dy}{dx}=(b+2x^2)\frac{d(a+x)}{dx}+(a+x)\frac{d(b+2x^2)}{dx}, \text{ by IV.,}$$
$$=(b+2x^2)+4(a+x)x$$
$$=b+6x^2+4ax.$$

3. $y=\frac{4x^3}{(a+x^2)^3}$.

Applying VI., and VII.,

$$\frac{dy}{dx}=\frac{(a+x^2)^3\times 12x^2-4x^3\times 3(a+x^2)^2\times(2x)}{(a+x^2)^6}$$
$$=\frac{12x^2(a-x^2)}{(a+x^2)^4}.$$

4. $y=x(1+x^2)(1+x^3)$.

$$\frac{dy}{dx}=(1+x^2)(1+x^3)+x(1+x^3)\frac{d}{dx}(1+x^2)+x(1+x^2)\frac{d}{dx}(1+x^3)$$
$$=(1+x^2)(1+x^3)+x(1+x^3)(2x)+x(1+x^2)(3x^2)$$
$$=1+3x^2+4x^3+6x^5.$$

5. $y=\frac{1+x}{1+x^2}$. $\qquad \frac{dy}{dx}=\frac{1-2x-x^2}{(1+x^2)^2}$.

6. $y=a\sqrt{x}$. $\qquad \frac{dy}{dx}=\frac{a}{2\sqrt{x}}$.

7. $y=(a+x)^m(b+x)^n$. $\qquad \frac{dy}{dx}=(a+x)^{m-1}(b+x)^{n-1}[m(b+x)+n(a+x)]$.

8. $y=\sqrt{\frac{1+x}{1-x}}$. $\qquad \frac{dy}{dx}=\frac{1}{(1-x)\sqrt{1-x^2}}$.

9. $y=\frac{1}{x^n}$. $\qquad \frac{dy}{dx}=-\frac{n}{x^{n+1}}$.

10. $y=(a-x)\sqrt{a+x}$. $\qquad \frac{dy}{dx}=-\frac{a+3x}{2\sqrt{a+x}}$.

11. $y = x(a^2 + x^2)\sqrt{a^2 - x^2}$. $\qquad \dfrac{dy}{dx} = \dfrac{a^4 + a^2x^2 - 4x^4}{\sqrt{a^2 - x^2}}$.

12. $y = \dfrac{x}{x + \sqrt{1 - x^2}}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2} + 2x(1 - x^2)}$.

13. $y = \dfrac{\sqrt{1 + x} + \sqrt{1 - x}}{\sqrt{1 + x} - \sqrt{1 - x}}$. $\qquad \dfrac{dy}{dx} = -\dfrac{1 + \sqrt{1 - x^2}}{x^2\sqrt{1 - x^2}}$.

14. $y = (1 - 3x^2 + 6x^4)(1 + x^2)^3$. $\qquad \dfrac{dy}{dx} = 60x^5(1 + x^2)^2$.

15. $y = \dfrac{3x^3 + 2}{x(x^3 + 1)^{\frac{2}{3}}}$. $\qquad \dfrac{dy}{dx} = -\dfrac{2}{x^2(x^3 + 1)^{\frac{5}{3}}}$.

16. $y = \dfrac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 + x^2} - \sqrt{1 - x^2}}$. $\qquad \dfrac{dy}{dx} = -\dfrac{2}{x^3}\left(1 + \dfrac{1}{\sqrt{1 - x^4}}\right)$.

17. $y = 3(x^2 + 1)^{\frac{4}{3}}(4x^2 - 3)$. $\qquad \dfrac{dy}{dx} = 56x^3(x^2 + 1)^{\frac{1}{3}}$.

18. $y = \dfrac{2x^2 - 1}{x\sqrt{1 + x^2}}$. $\qquad \dfrac{dy}{dx} = \dfrac{1 + 4x^2}{x^2(1 + x^2)^{\frac{3}{2}}}$.

19. $y = \dfrac{\sqrt{(x + a)^3}}{\sqrt{x - a}}$. $\qquad \dfrac{dy}{dx} = \dfrac{(x - 2a)\sqrt{x + a}}{(x - a)^{\frac{3}{2}}}$.

20. $y = \dfrac{1}{(a + x)^m}\,\dfrac{1}{(b + x)^n}$. $\qquad \dfrac{dy}{dx} = -\dfrac{m(b + x) + n(a + x)}{(a + x)^{m+1}(b + x)^{n+1}}$.

21. $y = \left(\dfrac{x}{1 + \sqrt{1 - x^2}}\right)^n$. $\qquad \dfrac{dy}{dx} = \dfrac{ny}{x\sqrt{1 - x^2}}$.

22. $y = \sqrt{\dfrac{1 - x^2}{(1 + x^2)^3}}$. $\qquad \dfrac{dy}{dx} = -\dfrac{2x(2 - x^2)}{(1 - x^2)^{\frac{1}{2}}(1 + x^2)^{\frac{5}{2}}}$.

CHAPTER III.

DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS.

ART. 16. DEFINITIONS.

All functions that are not algebraic are called *transcendental.* Transcendental functions are divided into four classes:

1st. Logarithmic functions; those in which a logarithm of a variable is involved.

2d. Exponential functions; those in which a variable enters as an exponent.

3d. Trigonometric functions; those involving sines, cosines, tangents, etc., in which the arc is the independent variable.

4th. Inverse trigonometric functions; those derived from trigonometric functions, by taking the arc as the dependent variable. Thus, from the trigonometric function, $y = \sin x$, is obtained the inverse function, $x = \text{arc} \sin y$, which is read, "x equals the arc whose sine is y." The inverse trigonometric functions are also called *circular functions* and *anti-trigonometric functions.*

The inverse trigonometric functions are often expressed differently, as shown in the following identities:

$$\text{arc} \sin y \equiv \sin^{-1} y; \quad \text{arc} \tan y \equiv \tan^{-1} y; \quad \text{arc cosec}\, y \equiv \text{cosec}^{-1} y.$$

This second notation, employed to express inverse trigonometric functions, was suggested by the use of negative exponents in algebra, but the student is cautioned against the error of regarding $\sin^{-1} y$ as equivalent to $\frac{1}{\sin y}$. Another application of this notation for inverse functions is seen in an anti-logarithm; if $y = \log x$, then $x = \log^{-1} y$.

ART. 17. BASE OF THE NATURAL SYSTEM OF LOGARITHMS.

The base of the natural system of logarithms is the limit of $\left(1+\frac{1}{x}\right)^x$ as x approaches infinity.

By the Binomial Theorem,

$$\left(1+\frac{1}{x}\right)^x = 1 + x\left(\frac{1}{x}\right) + \frac{x(x-1)}{1\cdot 2}\left(\frac{1}{x}\right)^2 + \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}\left(\frac{1}{x}\right)^3 + \cdots$$

$$= 1 + 1 + \frac{1-\frac{1}{x}}{1\cdot 2} + \frac{\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)}{1\cdot 2\cdot 3} + \cdots.$$

When x increases indefinitely,

$$\text{limit}\left(1+\frac{1}{x}\right)^x = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots.$$

This limit is usually denoted by e.

Therefore $$e = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots.$$

By summing this series the value of e is found to be 2.7182818+, which is the base of the natural system of logarithms.

ART. 18. LOGARITHMIC FUNCTIONS.

Throughout this chapter, v and u will always be regarded as functions of a single independent variable x.

Let the base of the system of logarithms be a;

then let $$y = \log_a v;$$

hence $$y + \Delta y = \log_a (v + \Delta v),$$

$$\Delta y = \log_a (v + \Delta v) - \log_a v$$

$$= \log_a \frac{(v + \Delta v)}{v} = \log_a \left(1 + \frac{\Delta v}{v}\right)$$

$$= \frac{\Delta v}{v} \log_a \left(1 + \frac{\Delta v}{v}\right)^{\frac{v}{\Delta v}};$$

and $$\frac{\Delta y}{\Delta x} = \frac{\frac{\Delta v}{\Delta x}}{v} \log_a \left(1 + \frac{\Delta v}{v}\right)^{\frac{v}{\Delta v}}.$$

Now as Δx approaches zero, Δv approaches zero; and therefore $\frac{\Delta v}{v}$ approaches zero and $\frac{v}{\Delta v}$ approaches infinity.

If $\frac{v}{\Delta v}$ be substituted for x in the preceding article, the limit of $\left(1 + \frac{\Delta v}{v}\right)^{\frac{v}{\Delta v}}$ is seen to be e.

Therefore $$\frac{dy}{dx} = \log_a e \frac{\frac{dv}{dx}}{v}. \qquad \text{VIII.}$$

$\text{Log}_a e$ is the modulus of the system in which the logarithm is taken and may be denoted by M.

VIII. Hence, *the derivative of the logarithm of a function of x is equal to the modulus, multiplied by the derivative of the function, divided by the function.*

Hereafter, when no base is specified, it will be understood that natural logarithms are used; then

$$M = \log_a e = \log_e e = 1,$$

and VIII. becomes $$\frac{dy}{dx} = \frac{dv}{dx}\frac{1}{v}. \qquad \text{VIII. } a.$$

ART. 19. THE EXPONENTIAL FUNCTION WITH A CONSTANT BASE.

Let the exponential function with a constant base be

$$y = a^v.$$

Taking the logarithm of each member,

$$\log y = v \log a.$$

Differentiating by VIII.,

$$M\frac{\frac{dy}{dx}}{y} = \log a \frac{dv}{dx};$$

therefore
$$\frac{dy}{dx} = \frac{a^v \log a}{M} \frac{dv}{dx}. \qquad \text{IX.}$$

And when natural logarithms are used,

$$\frac{dy}{dx} = a^v \log a \frac{dv}{dx}. \qquad \text{IX. } a.$$

If $a = e$ in IX. a, since $\log_e e = 1$,

$$\frac{dy}{dx} = e^v \frac{dv}{dx}. \qquad \text{IX. } b.$$

If $v = x$ in IX. a,
$$\frac{dy}{dx} = a^x \log a. \qquad (1)$$

If $a = e$ in (1),
$$\frac{dy}{dx} = e^x. \qquad \text{IX. } c.$$

IX. Hence, *the derivative of an exponential function with a constant base is equal to the function multiplied by the logarithm of the base and by the derivative of the exponent, divided by the modulus.*

Art. 20. The Exponential Function with a Variable Base.

Let the exponential function with a variable base be

$$y = u^v.$$

Then
$$\log y = v \log u;$$

and by VIII.,
$$M\frac{\frac{dy}{dx}}{y} = M\frac{v\frac{du}{dx}}{u} + \log u \frac{dv}{dx}.$$

Therefore
$$\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + \frac{u^v \log u}{M}\frac{dv}{dx}. \qquad \text{X.}$$

X. Hence, *the derivative of an exponential function with a variable base is equal to the sum of two derivatives; the first being obtained as though the base were variable and the exponent constant, and the secona as though the base were constant and the exponent variable.*

PROBLEMS.

1. $y = x^2 \log x.$ $\qquad \frac{dy}{dx} = x(2\log x + 1).$

2. $y = \log(2x + x^3).$ $\qquad \frac{dy}{dx} = \frac{2 + 3x^2}{2x + x^3}.$

3. $y = \log \frac{a + x}{a - x}.$ $\qquad \frac{dy}{dx} = \frac{2a}{a^2 - x^2}.$

4. $y = \log \sqrt{\frac{1 + x}{1 - x}}.$ $\qquad \frac{dy}{dx} = \frac{1}{1 - x^2}.$

5. $y = \log(x + \sqrt{1 + x^2}).$ $\qquad \frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}.$

6. $y = \log x^2.$ $\qquad \frac{dy}{dx} = \frac{2}{x}.$

7. $y = \log^2 x.$ $\qquad \frac{dy}{dx} = \frac{2\log x}{x}.$

8. $y = \log(\log x).$ $\qquad \frac{dy}{dx} = \frac{1}{x\log x}.$

9. $y = \log \frac{\sqrt{x^2 + 1} - x}{\sqrt{x^2 + 1} + x}.$ $\qquad \frac{dy}{dx} = -\frac{2}{\sqrt{x^2 + 1}}.$

10. $y = \log \frac{\sqrt{1 + x} + \sqrt{1 - x}}{\sqrt{1 + x} - \sqrt{1 - x}}.$ $\qquad \frac{dy}{dx} = -\frac{1}{x\sqrt{1 - x^2}}.$

11. $y = \log(\sqrt{1 + x^2} + \sqrt{1 - x^2}).$ $\qquad \frac{dy}{dx} = \frac{1}{x} - \frac{1}{x\sqrt{1 - x^4}}.$

12. $y = a^{e^x}.$ $\qquad \frac{dy}{dx} = a^{e^x} e^x \log a.$

13. $y = a^{x^2}.$ $\qquad \frac{dy}{dx} = 2a^{x^2} \cdot \log a \cdot x.$

14. $y = e^x(x - 1).$ $\qquad \frac{dy}{dx} = e^x x.$

15. $y = 2e^{\sqrt{x}}(x^{\frac{3}{2}} - 3x + 6x^{\frac{1}{2}} - 6).$ $\qquad \frac{dy}{dx} = xe^{\sqrt{x}}.$

16. $y=\dfrac{e^x-1}{e^x+1}$. $\qquad \dfrac{dy}{dx}=\dfrac{2e^x}{(e^x+1)^2}$.

17. $y=\dfrac{e^x}{1+x}$. $\qquad \dfrac{dy}{dx}=\dfrac{xe^x}{(1+x)^2}$.

18. $y=x^{\frac{1}{x}}$. $\qquad \dfrac{dy}{dx}=\dfrac{x^{\frac{1}{x}}(1-\log x)}{x^2}$.

19. $y=x^n a^x$. $\qquad \dfrac{dy}{dx}=a^x x^{n-1}(n+x\log a)$.

20. $y=x^x$. $\qquad \dfrac{dy}{dx}=x^x(\log x+1)$.

21. $y=x^{\log x}$. $\qquad \dfrac{dy}{dx}=\log x^2\cdot x^{\log x-1}$.

22. $y=x^{x^x}$. $\qquad \dfrac{dy}{dx}=x^{x^x}\left(\log x+\log^2 x+\dfrac{1}{x}\right)x^x$.

23. $y=e^x[x^n-nx^{n-1}+n(n-1)x^{n-2}-\cdots]$. $\qquad \dfrac{dy}{dx}=e^x x^n$.

Art. 21. Circular Measure.*

In higher mathematics, angles are not measured by the ordinary degree or gradual system, but in terms of another unit. The *circular measure* of an arc of a circle is the ratio of the length of the arc to the length of its radius; and it is evident that this ratio does not vary with the radius. Thus the value of an arc of 360° in circular measure is $\dfrac{2\pi r}{r}=2\pi$, of 180° is π, of 90° is $\dfrac{\pi}{2}$, and of 1° is $\dfrac{\pi}{180}$.

The angle at the centre of a circle subtended by an arc equal to the radius is the *radian* or circular unit.

Let x denote the number of degrees in an angle, and z the number of radians in the same angle; then since there are π radians in two right angles,

$$\frac{x}{180}=\frac{z}{\pi}.$$

Therefore $$z=\frac{\pi}{180}x,$$

* Hall's *Mensuration*, §§ 9, 10, 11, and 12.

and $$x = \frac{180}{\pi} z.$$

Hence, to reduce from gradual to circular measure, the number of degrees in the angle is multiplied by $\frac{\pi}{180}$; and to reduce from circular to gradual measure, the circular value is multiplied by $\frac{180}{\pi}$.

ART. 22. LIMITING VALUE OF $\frac{\sin\theta}{\theta}$.

Let the small angle AOB in Fig. 5a be represented by θ, and the radius OA by a; and let BC, AB and AD be $\sin\theta$, chord θ and $\tan\theta$, respectively.

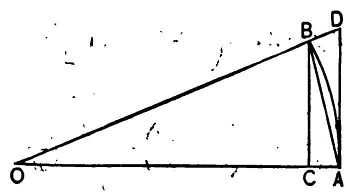

FIG. 5a.

The area of the triangle $AOB = \frac{1}{2} a^2 \sin\theta$;

The area of the sector $AOB = \frac{1}{2} a^2\theta$;

The area of the triangle $AOD = \frac{1}{2} a^2 \tan\theta$;

and these areas are obviously in an ascending order of magnitude;

hence $$\tan\theta > \theta > \sin\theta,$$

or $$\frac{\tan\theta}{\sin\theta} > \frac{\theta}{\sin\theta} > 1.$$

Thus $\frac{\theta}{\sin\theta}$ lies between $\frac{\tan\theta}{\sin\theta}$ and 1; but when θ approaches zero, $\frac{\tan\theta}{\sin\theta}$ or $\frac{1}{\cos\theta}$ approaches 1; hence, as θ diminishes indefinitely, $\frac{\theta}{\sin\theta}$ approaches the limit unity.

ART. 23. TRIGONOMETRIC FUNCTIONS.

1. Differentiation of $\sin v$.

Let $$y = \sin v,$$

then $$y + \Delta y = \sin(v + \Delta v);$$

therefore $$\Delta y = \sin(v + \Delta v) - \sin v.$$

By Trigonometry,

$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A + B)\sin\tfrac{1}{2}(A - B).$$

Substituting $v + \Delta v = A$, and $v = B$, in this formula,

$$\Delta y = 2\cos\left(v + \frac{\Delta v}{2}\right)\sin\frac{\Delta v}{2};$$

hence $$\frac{\Delta y}{\Delta x} = \cos\left(v + \frac{\Delta v}{2}\right)\frac{\sin\frac{\Delta v}{2}}{\frac{\Delta v}{2}}\frac{\Delta v}{\Delta x}.$$

When Δx approaches zero, Δv approaches zero, and by Art. 23,

$$\text{limit } \frac{\sin\frac{\Delta v}{2}}{\frac{\Delta v}{2}} \text{ is unity.}$$

Therefore $$\frac{dy}{dx} = \cos v\frac{dv}{dx}. \qquad \text{XI.}$$

XI. Hence, *the derivative of $\sin v$ is equal to $\cos v$ multiplied by* $\frac{dv}{dx}$.

2. Differentiation of $\cos v$.

Let $$y = \cos v,$$

then $$\Delta y = \cos(v + \Delta v) - \cos v.$$

By Trigonometry,

$$\cos A - \cos B = -2\sin\tfrac{1}{2}(A + B)\sin\tfrac{1}{2}(A - B).$$

Substituting $v + \Delta v = A$, and $v = B$,

$$\Delta y = -2 \sin\left(v + \frac{\Delta v}{2}\right) \sin \frac{\Delta v}{2};$$

hence
$$\frac{\Delta y}{\Delta x} = -\sin\left(v + \frac{\Delta v}{2}\right) \frac{\sin \frac{\Delta v}{2}}{\frac{\Delta v}{2}} \frac{\Delta v}{\Delta x}.$$

Therefore
$$\frac{dy}{dx} = -\sin v \frac{dv}{dx}. \qquad \text{XII.}$$

XII. Hence, *the derivative of* $\cos v$ *is negative, and equal to* $\sin v$ *multiplied by* $\frac{dv}{dx}$.

3. Differentiation of $\tan v$.

Let
$$y = \tan v = \frac{\sin v}{\cos v}.$$

By VI.,
$$\frac{d}{dx}\left(\frac{\sin v}{\cos v}\right) = \frac{\cos v \frac{d}{dx}(\sin v) - \sin v \frac{d}{dx}(\cos v)}{\cos^2 v}$$

$$= \frac{\cos^2 v \frac{dv}{dx} + \sin^2 v \frac{dv}{dx}}{\cos^2 v}$$

$$= \sec^2 v \frac{dv}{dx}. \qquad \text{XIII.}$$

XIII. Hence, *the derivative of* $\tan v$ *is equal to* $\sec^2 v$ *multiplied by* $\frac{dv}{dx}$.

4. Differentiation of $\text{cotan}\, v$.

Let
$$y = \text{cotan}\, v = \frac{\cos v}{\sin v}.$$

By VI.,
$$\frac{d}{dx}\left(\frac{\cos v}{\sin v}\right) = \frac{\sin v\left(-\sin v \frac{dv}{dx}\right) - \cos v\left(\cos v \frac{dv}{dx}\right)}{\sin^2 v}$$

$$= -\text{cosec}^2 v \frac{dv}{dx}. \qquad \text{XIV.}$$

XIV. Hence, *the derivative of* $\text{cotan}\, v$ *is negative, and equal to* $\text{cosec}^2 v$ *multiplied by* $\frac{dv}{dx}$.

5. Differentiation of $\sec v$.

Let $$y = \sec v = \frac{1}{\cos v}.$$

By Art. 14, Cor. 1,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{\cos v}\right) = -\frac{\frac{d}{dx}(\cos v)}{\cos^2 v} = \frac{\sin v \frac{dv}{dx}}{\cos^2 v}$$

$$= \sec v \tan v \frac{dv}{dx}. \qquad \text{XV.}$$

XV. Hence, *the derivative of* $\sec v$ *is equal to* $\sec v$, *multiplied by* $\tan v$, *into* $\frac{dv}{dx}$.

6. Differentiation of $\operatorname{cosec} v$.

Let $$y = \operatorname{cosec} v = \frac{1}{\sin v}.$$

By Art. 14, Cor. 1,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{\sin v}\right) = -\frac{\cos v \frac{dv}{dx}}{\sin^2 v}$$

$$= -\operatorname{cosec} v \operatorname{cotan} v \frac{dv}{dx}. \qquad \text{XVI.}$$

XVI. Hence, *the derivative of* $\operatorname{cosec} v$ *is negative, and equal to* $\operatorname{cosec} v$, *multiplied by* $\operatorname{cotan} v$, *into* $\frac{dv}{dx}$.

7. Differentiation of $\operatorname{vers} v$.

Let $$y = \operatorname{vers} v = 1 - \cos v.$$

Then $$\frac{dy}{dx} = \frac{d}{dx}(1 - \cos v) = \sin v \frac{dv}{dx}. \qquad \text{XVII.}$$

XVII. Hence, *the derivative of* $\operatorname{vers} v$ *is equal to* $\sin v$ *into* $\frac{dv}{dx}$.

8. Differentiation of $\operatorname{covers} v$.

Let $$y = \operatorname{covers} v = 1 - \sin v.$$

Then $$\frac{dy}{dx} = \frac{d}{dx}(1 - \sin v) = -\cos v \frac{dv}{dx}. \qquad \text{XVIII.}$$

XVIII. Hence, *the derivative of covers v is negative, and equal to cos v into* $\frac{dv}{dx}$.

ART. 24. INVERSE TRIGONOMETRIC FUNCTIONS.

It should be remembered that there are two ways of indicating the inverse trigonometric functions. The functions arc $\sin x$, arc $\cos x$, arc $\tan x$, etc., are often written as follows: $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, etc., respectively.

1. Differentiation of $y = \text{arc} \sin v$.

Then $$v = \sin y.$$

By XI., $$\frac{dv}{dx} = \cos y \frac{dy}{dx};$$

hence $$\frac{dy}{dx} = \frac{1}{\cos y}\frac{dv}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}}\frac{dv}{dx}.$$

Therefore $$\frac{d(\text{arc} \sin v)}{dx} = \frac{\frac{dv}{dx}}{\sqrt{1 - v^2}}. \qquad \text{XIX.}$$

2. Differentiation of $y = \text{arc} \cos v$.

Then $$v = \cos y.$$

By XII., $$\frac{dv}{dx} = -\sin y \frac{dy}{dx};$$

hence $$\frac{dy}{dx} = -\frac{1}{\sin y}\frac{dv}{dx} = -\frac{1}{\sqrt{1 - \cos^2 y}}\frac{dv}{dx}.$$

Therefore $$\frac{d(\text{arc} \cos v)}{dx} = -\frac{\frac{dv}{dx}}{\sqrt{1 - v^2}}. \qquad \text{XX.}$$

3. Differentiation of $y = \text{arc} \tan v$.

Then $$v = \tan y.$$

By XIII., $$\frac{dv}{dx} = \sec^2 y \frac{dy}{dx};$$

hence $$\frac{dy}{dx} = \frac{1}{\sec^2 y}\frac{dv}{dx} = \frac{1}{1 + \tan^2 y}\frac{dv}{dx}.$$

Therefore $$\frac{d\,(\text{arc tan } v)}{dx} = \frac{\frac{dv}{dx}}{1+v^2}. \qquad \text{XXI.}$$

4. Differentiation of $y = \text{arc cot } v$.

Then $$v = \cot y.$$

By XIV., $$\frac{dv}{dx} = -\operatorname{cosec}^2 y \frac{dy}{dx};$$

hence $$\frac{dy}{dx} = -\frac{1}{\operatorname{cosec}^2 y}\frac{dv}{dx} = -\frac{1}{1+\cot^2 y}\frac{dv}{dx}.$$

Therefore $$\frac{d\,(\text{arc cot } v)}{dx} = -\frac{\frac{dv}{dx}}{1+v^2}. \qquad \text{XXII.}$$

5. Differentiation of $y = \text{arc sec } v$.

Then $$v = \sec y.$$

By XV., $$\frac{dv}{dx} = \sec y \tan y \frac{dy}{dx};$$

hence $$\frac{dy}{dx} = \frac{1}{\sec y \tan y}\frac{dv}{dx} = \frac{1}{\sec y\sqrt{\sec^2 y - 1}}\frac{dv}{dx}.$$

Therefore $$\frac{d\,(\text{arc sec } v)}{dx} = \frac{\frac{dv}{dx}}{v\sqrt{v^2-1}}. \qquad \text{XXIII.}$$

6. Differentiation of $y = \text{arc cosec } v$.

Then $$v = \operatorname{cosec} y.$$

By XVI., $$\frac{dv}{dx} = -\operatorname{cosec} y \cot y \frac{dy}{dx};$$

hence $$\frac{dy}{dx} = -\frac{1}{\operatorname{cosec} y \cot y}\frac{dv}{dx} = -\frac{1}{\operatorname{cosec} y\sqrt{\operatorname{cosec}^2 y - 1}}\frac{dv}{dx}.$$

Therefore $$\frac{d\,(\text{arc cosec } v)}{dx} = -\frac{\frac{dv}{dx}}{v\sqrt{v^2-1}}. \qquad \text{XXIV.}$$

7. Differentiation of $y = \text{arc vers}\, v$.

Then $$v = \text{vers}\, y.$$

By XVII., $$\frac{dv}{dx} = \sin y \frac{dy}{dx} = \sqrt{1 - \cos^2 y}\, \frac{dy}{dx};$$

hence $$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \cos^2 y}} \frac{dv}{dx} = \frac{1}{\sqrt{2 \,\text{vers}\, y - \text{vers}^2 y}} \frac{dv}{dx}.$$

Therefore $$\frac{d\,(\text{arc vers}\, v)}{dx} = \frac{\frac{dv}{dx}}{\sqrt{2v - v^2}}. \qquad \text{XXV.}$$

8. Differentiation of $y = \text{arc covers}\, v$

Then $$v = \text{covers}\, y.$$

By XVIII., $$\frac{dv}{dx} = -\cos y \frac{dy}{dx} = -\sqrt{1 - \sin^2 y}\, \frac{dy}{dx};$$

hence $$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - \sin^2 y}} \frac{dv}{dx} = -\frac{1}{\sqrt{2 \,\text{covers}\, y - \text{covers}^2 y}} \frac{dv}{dx}.$$

Therefore $$\frac{d\,(\text{arc covers}\, v)}{dx} = -\frac{\frac{dv}{dx}}{\sqrt{2v - v^2}}. \qquad \text{XXVI.}$$

PROBLEMS.

1. $y = \sin nx$. $\qquad \dfrac{dy}{dx} = n \cos nx.$

2. $y = \sin^n x$. $\qquad \dfrac{dy}{dx} = n \sin^{n-1} x \cos x.$

3. $y = \cot^2 (x^3)$. $\qquad \dfrac{dy}{dx} = -6x^2 \cot (x^3) \operatorname{cosec}^2 (x^3)$

4. $y = \log (\sin^2 x)$. $\qquad \dfrac{dy}{dx} = 2 \cot x.$

5. $y = \sin 2x \cos x$. $\qquad \dfrac{dy}{dx} = 2 \cos 2x \cos x - \sin 2x \sin x.$

6. $y = e^x \cos x$. $\qquad \dfrac{dy}{dx} = e^x (\cos x - \sin x).$

7. $y = e^{\cos x}\sin x.$ $\qquad \dfrac{dy}{dx} = e^{\cos x}(\cos x - \sin^2 x).$

8. $y = \sin\log x.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{x}\cos(\log x).$

9. $y = (\cos x)^{\sin x}.$ $\qquad \dfrac{dy}{dx} = (\cos x)^{\sin x}[\cos x \log\cos x - \sin x \tan x].$

10. $y = \log\tan x.$ $\qquad \dfrac{dy}{dx} = \dfrac{2}{\sin 2x}.$

11. $y = \log\sqrt{\dfrac{1+\sin x}{1-\sin x}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{\cos x}.$

12. $y = \log\sec x.$ $\qquad \dfrac{dy}{dx} = \tan x.$

13. $y = \operatorname{arc}\sin\dfrac{x}{a}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{a^2 - x^2}}.$

14. $y = \operatorname{arc}\tan\dfrac{x}{a}.$ $\qquad \dfrac{dy}{dx} = \dfrac{a}{a^2 + x^2}.$

15. $y = \operatorname{arc}\sin - \dfrac{2}{1+x^2}.$ $\qquad \dfrac{dy}{dx} = \dfrac{2}{1+x^2}.$

16. $y = \operatorname{arc}\sin(3x - 4x^3).$ $\qquad \dfrac{dy}{dx} = \dfrac{3}{\sqrt{1-x^2}}.$

17. $y = \operatorname{arc}\sec 2x.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{x\sqrt{4x^2 - 1}}.$

18. $y = \operatorname{arc}\tan\dfrac{2x}{1-x^2}.$ $\qquad \dfrac{dy}{dx} = \dfrac{2}{1+x^2}.$

19. $y = x\sqrt{a^2 - x^2} + a^2 \operatorname{arc}\sin\dfrac{x}{a}.$ $\qquad \dfrac{dy}{dx} = 2\sqrt{a^2 - x^2}.$

20. $y = e^{\operatorname{arc}\tan x}.$ $\qquad \dfrac{dy}{dx} = \dfrac{e^{\operatorname{arc}\tan x}}{1+x^2}.$

21. $y = \sqrt{a^2 - x^2} + a\operatorname{arc}\sin\dfrac{x}{a}.$ $\qquad \dfrac{dy}{dx} = \left(\dfrac{a-x}{a+x}\right)^{\frac{1}{2}}.$

22. $y = \operatorname{arc}\tan\dfrac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{2}.$

23. $y = \text{arc cot}\,\frac{a}{x} + \log\sqrt{\frac{x-a}{x+a}}.$ $\qquad \frac{dy}{dx} = \frac{2ax^2}{x^4 - a^4}.$

24. $y = \frac{x(\text{arc sin}\,x)}{\sqrt{1-x^2}} + \log\sqrt{1-x^2}.$ $\qquad \frac{dy}{dx} = \frac{\text{arc sin}\,x}{(1-x^2)^{\frac{3}{2}}}.$

25. $y = \frac{1}{6}\log\frac{(x+1)^3}{x^3+1} + \frac{1}{\sqrt{3}}\text{arc tan}\frac{2x-1}{\sqrt{3}}.$ $\qquad \frac{dy}{dx} = \frac{1}{x^3+1}.$

26. $y = \frac{e^{ax}(a\sin x - \cos x)}{a^2+1}.$ $\qquad \frac{dy}{dx} = e^{ax}\sin x.$

27. $y = \text{arc sec}\frac{x\sqrt{5}}{2\sqrt{x^2+x-1}}.$ $\qquad \frac{dy}{dx} = \frac{1}{x\sqrt{x^2+x-1}}.$

28. $y = \log\sqrt[4]{\frac{1+x}{1-x}} + \frac{1}{2}\text{arc tan}\,x.$ $\qquad \frac{dy}{dx} = \frac{1}{1-x^4}.$

D

CHAPTER IV.

DIFFERENTIALS AND RATES.

ART. 25. DEFINITION.

The formulas for differentiation given in the preceding chapters have been established by the *method of limits*. In this chapter another method of treatment will be presented which is called the *method of infinitesimals*. Here dx is regarded as an *infinitesimal* or as a variable which approaches the limit zero. According to this second method, the independent variable is supposed to change by the continued addition of an infinitely small constant increment. This increment is called the *differential of the variable*, and the corresponding increment of the function is called the *differential of the function*. The differential of a variable may then be defined as the difference between two consecutive values of the variable. Hitherto, the symbol $\frac{dy}{dx}$ has been regarded as a whole, but here it is defined as the ratio of the differential of the function to the differential of the independent variable, and is regarded as a fraction. The phraseology and notation of the two methods are different, but they give identical results. To illustrate:

Let $$y = x^5,$$

then by VII., $$\frac{dy}{dx} = 5x^4.$$

If differentials are used, the equation becomes

$$dy = 5x^4dx,$$

which would be read, "The differential of y is equal to $5x^4$ times the differential of x."

In general, let $$\frac{dy}{dx} = f'(x),$$

then $$dy = f'(x)dx.$$

Now the reason for sometimes calling the derivative the differential

coefficient is apparent, as it is seen to be the coefficient of dx in the differential of $f(x)$.

If each member of each of the formulas, I.–XXVI., be multiplied by dx, a corresponding set of formulas will be obtained for the differentials of functions.* Every formula for differentiation as previously obtained can therefore be transformed into a *differential formula.*

In the function,

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

Δy denotes the change in the dependent variable corresponding to a change Δx in the independent variable, and the quotient shows the average rate of change in y for the interval Δx.

As $$\Delta x \doteq 0, \lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right] = \frac{dy}{dx},$$

and the derivative measures the rate of change of the function.

In Fig. 6, PT is the tangent to the curve at P, $PM = \Delta x$ and $MP' = \Delta y$ are the increments of the variables in passing from P to P'. If the points P and P' are consecutive, the differentials are $PM = dx$ and $MN = dy$.

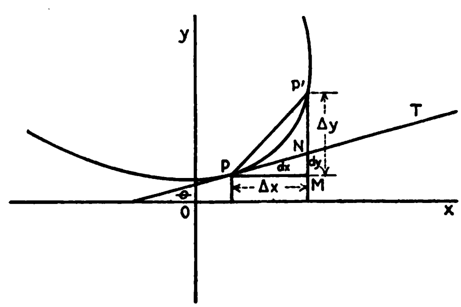

FIG. 6.

A point moving along the curve at the point P is moving in the direction of the tangent, and the differential, dy, is the amount that y would increase when x increases if the direction of motion was unchanged.

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* Lagrange, in *Mécanique Analytique*, says: "When we have properly conceived the spirit of the infinitesimal method, and are convinced of the exactness of the results by the geometrical method of prime or ultimate ratios, or by the analytical method of derived functions, we may employ infinitely small quantities as a sure and valuable means of abridging and simplifying our demonstrations."

The student will be much benefited by plotting curves whose equations are of the form $y = f(x)$, and interpreting the derivative obtained from each equation.

So, $$\frac{dy}{dx} = \tan\theta = f'(x),$$

and $$dy = f'(x)dx,$$

and the derivative measures the *rate of change* at a point.

It should be noted that dy and Δy are equal only when $y = f(x)$ is a straight line. But it is evident that as Δx diminishes, dy and Δy approach each other, that is,

$$\lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{dy}\right] = 1.$$

If a moving particle passes over a distance s in the time t, t may be taken as the independent variable and s as the dependent variable. Then Δs is the distance described in the interval Δt, and $\frac{\Delta s}{\Delta t}$ is the average rate of change during the interval.

The actual rate at any instant is,

$$\lim_{\Delta t \doteq 0}\left[\frac{\Delta s}{\Delta t}\right] = \frac{ds}{dt}.$$

If y represents any quantity that varies with the time, then

$$\lim_{\Delta t \doteq 0}\left[\frac{\Delta y}{\Delta t}\right] = \frac{dy}{dt},$$

will be the rate of change in y at any instant.

Since $$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}, \tag{1}$$

$\frac{dy}{dx}$ can be taken to represent the ratio of the rates of change of y and x.

Instead of writing the time derivatives $\frac{dy}{dt}$ and $\frac{dx}{dt}$ in (1) for the rates, it is convenient simply to use dy and dx.

ART. 26. GEOMETRIC DERIVATION OF THE FORMULAS FOR THE DIFFERENTIATION OF THE TRIGONOMETRIC FUNCTIONS.

In Fig. 6A, let AP represent a circular arc x, with radius $= 1$, and $PP' = dx$, an infinitely small increment given to x. PS is drawn parallel to OA, and PN and $P'M$ are consecutive ordinates.

$PN = \sin x$; $P'M = \sin(x + dx)$; therefore $P'S = d \sin x$.

$ON = \cos x$; $OM = \cos(x + dx)$; therefore $NM = - d \cos x$.

The triangle $PP'S$ is a right triangle, and $\angle PP'S = \angle PON$.

Hence, $d \sin x = P'S = PP' \cos PP'S = \cos x\, dx,$ XI.

and $d \cos x = - MN = - PP' \sin PP'S$

$= - \sin x\, dx.$ XII.

$AT = \tan x,$

and $AT' = \tan (x + dx);$

hence $TT' = d \tan x,$

and $CT' = d \sec x.$

$BD = \cot x,$

and $BH = \cot (x + dx);$

hence $HD = - d \cot x,$

and $HE = - d \operatorname{cosec} x.$

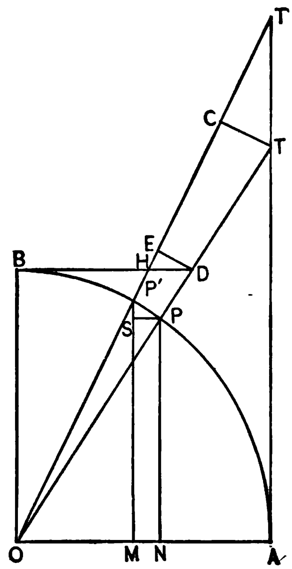

FIG. 6A.

From the triangles CTT' and HDE, similar to NOP, the differentials of the remaining trigonometric functions may be obtained.

It will be noticed in this article, that the differential of a function is negative when the function decreases as the independent variable increases.

PROBLEMS.

1. If the side of an equilateral triangle increases uniformly at the rate of 2 inches per second, at what rate does the altitude increase?

Let x = a side of the triangle, and y its altitude; then $y^2 = \frac{3}{4}x^2$. Differentiating, and solving for dy, gives $dy = \frac{\sqrt{3}}{2} dx$, which shows that if an infinitely small increment is given to x, the corresponding increment of y is $\frac{\sqrt{3}}{2}$ times as great; that is, the altitude increases $\frac{\sqrt{3}}{2}$ times

as fast as the side. When x is increasing at the rate of 2 inches per second, y is increasing at the rate of $\frac{\sqrt{3}}{2}$ times 2, or $\sqrt{3}$ inches per second.

REMARK. In these examples the differentials are regarded as rates. The rate of change of a variable at a given instant may be here defined as the increment which it would receive in a unit of time, if its change should be uniform throughout the interval. Thus, when a variable at a given instant is said to change at the rate of 2 inches per second, the meaning is, that an increment of 2 inches would be added in one second, if the change should continue uniform for one second.

2. If a circular plate of metal is expanded by heat, how rapidly does the area increase, when the radius is 2 inches long and increases at the rate of .01 inch per second?

Let $x =$ radius, and $y =$ area; then $y = \pi x^2$, and $dy = 2\pi x\,dx$.

When $x = 2$ inches, and $dx = .01$ inch per second, $dy = 0.4\pi$ square inches per second, which is the rate at which the area increases.

3. The common logarithm of 1174 is 3.069668. What is the logarithm of 1174.8, if the logarithm is assumed to change uniformly with the number?

Let $\qquad x =$ the number, and $y =$ its logarithm;

then $\qquad y = \log x$, and $dy = \frac{m}{x}\,dx$.

Hence, the increment of the logarithm is $\frac{m}{x}$ times as great as the increment of the number.

Therefore $\qquad dy = \frac{.43429448}{1174} \times .8 = .000295....$

And $\quad \log 1174.8 = 3.069668 + .000295 = 3.069963.$

REMARK. It will be seen from the equation $dy = \frac{m}{x}\,dx$, that as the number increases by equal constant increments, the logarithm will increase more and more slowly. So the assumption made in the last example is not strictly true, but for comparatively small changes in the number, the results are sufficiently accurate for practical applications.

The use of the Tabular Differences in tables of logarithms is based on this assumption.

4. In the parabola $y^2 = 12x$, find the point at which the ordinate and abscissa are increasing equally. *Ans.* The point (3.6).

5. At what part of the quadrant does the arc increase twice as rapidly as its sine? *Ans.* At 60°.

6. The logarithmic sine of 30° 5′ is 9.700062. What is the logarithmic sine of 30° 6′? *Ans.* 9.700280.

7. A boy is running on a horizontal plane directly towards the foot of a tower at the rate of 5 miles per hour. At what rate is he approaching the top when he is 60 feet from the base, the tower being 80 feet high? *Ans.* 3 miles per hour.

8. A vessel is sailing northwest at the rate of 10 miles per hour. At what rate is she making north latitude?

Ans. 7.07 + miles per hour.

9. If the diameter of a soap bubble is increasing at the rate of $\frac{1}{10}$ inch per second; when it is 5 inches, at what rate is the inclosed volume increasing?

10. A point is moving in the straight line, $4y - 3x = -6$ with a velocity of 5 feet per second. Find the x and y components of the velocity.

11. A light is suspended at the height of 12 feet above a straight horizontal pavement. If a boy 5 feet in height is walking away from the light at the rate of 140 feet per minute, how fast is his shadow lengthening?

12. If an error of 1% has been made in the measurement of the radius of a circle, what will be the error in the area?

13. A crank pin moves in a circle 4 feet in diameter at a constant velocity of 20 feet per second. What is the vertical component of the velocity when the crank angle with the horizontal is 30°?

14. A ship is anchored in water 18 feet deep, and the anchor chain is wound around a capstan which is 6 feet above the surface of the water. How fast is the ship moving if the chain is wound in at the rate of 3 feet a second when there are 30 feet of chain out?

15. A bundle is being elevated by a rope passing over a pulley 25 feet above the ground by a man holding the end of the rope and walking away horizontally in a straight line. How rapidly will the bundle ascend at the start if the rope is 50 feet long, the man's hands are 5 feet above the ground and he moves 10 feet a second?

16. A cistern is in the shape of an inverted right circular cone, with the diameter of its base and its height each being 10 feet. How fast is water being poured in when its depth is 5 feet and it is rising 4 inches a minute?

CHAPTER V.

INTEGRATION.

ART. 27. DEFINITION.

Integration is the operation of finding the function from which a given differential has been obtained. The result of the integration is called the *integral of the differential.* The symbol which indicates the operation of integration is $\int$. Since differentiation and integration are inverse operations, the symbols d and $\int$, as signs of operations, neutralize each other.*

The process of integration is of a tentative nature, depending on a previous knowledge of differentiation; just as division in arithmetic is a tentative process depending on a previous knowledge of multiplication.

For example, $$d(x^4) = 4x^3dx;$$

therefore $$\int 4x^3dx = x^4.$$

Again, $$d \sin x = \cos x\,dx,$$

hence, $$\int \cos x\,dx = \sin x.$$

Or, in general,

$$\frac{d[f(x)]}{dx} = f'(x),$$

$$d[f(x)] = f'(x)dx,$$

$$\int d\,[f(x)] = \int f'(x)dx,$$

$$f(x) = \int f'(x)dx.$$

Also, $$d\int f'(x)dx = f'(x)dx.$$

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* The symbol $\int$ is derived from the initial of the word "summation." Leibnitz introduced the letter S to denote the operation, and this gradually became elongated into the symbol $\int$.

Art. 28. Fundamental Formulas of Integration.

Differentiation is a direct process. The differential of any elementary function may be found by the application of a particular formula. In order to find the integral of a differential function, it is necessary to find the function which differentiated produces it, and this operation is a trial process. It is not possible in every case to obtain the integral.

The fundamental formulas of integration are obtained from the formulas of differentiation. In order to integrate a differential function it must be made to conform exactly to one of these formulas. If it is not apparent on inspection what function when differentiated produces it, the differential function must be transformed into some equivalent expression whose integral is given by one of the fundamental formulas. All integrations must ultimately be performed by the following formulas:

1.	$\int (du + dv - dw) = u + v - w;$	from	I.
2.	$\int a\,dv = av;$	"	II.
3.	$\int nav^{n-1}dv = av^n;$	"	VII.
4.	$\int \frac{a\,dv}{v} = a \log v;$	"	VIII. *a*.
5.	$\int a^v \log a\,dv = a^v;$	"	IX. *a*.
6.	$\int e^v dv = e^v;$	"	IX. *b*.
7.	$\int \cos v\,dv = \sin v;$	"	XI.
8.	$\int - \sin v\,dv = \cos v;$	"	XII.
9.	$\int \sec^2 v\,dv = \tan v;$	"	XIII.
10.	$\int - \operatorname{cosec}^2 v\,dv = \cot v;$	"	XIV.
11.	$\int \sec v \tan v\,dv = \sec v;$	"	XV.
12.	$\int - \operatorname{cosec} v \cot v\,dv = \operatorname{cosec} v;$	"	XVI.

13. $\int \sin v\, dv = \text{vers}\, v;$ from XVII.

14. $\int - \cos v\, dv = \text{covers}\, v;$ " XVIII.

15. $\int \frac{dv}{\sqrt{1-v^2}} = \text{arc} \sin v;$ " XIX.

16. $\int - \frac{dv}{\sqrt{1-v^2}} = \text{arc} \cos v;$ " XX.

17. $\int \frac{dv}{1+v^2} = \text{arc} \tan v;$ " XXI.

18. $\int - \frac{dv}{1+v^2} = \text{arc} \cot v;$ " XXII.

19. $\int \frac{dv}{v\sqrt{v^2-1}} = \text{arc} \sec v;$ " XXIII.

20. $\int - \frac{dv}{v\sqrt{v^2-1}} = \text{arc cosec}\, v;$ " XXIV.

21. $\int \frac{dv}{\sqrt{2v-v^2}} = \text{arc vers}\, v;$ " XXV.

22. $\int - \frac{dv}{\sqrt{2v-v^2}} = \text{arc covers}\, v;$ " XXVI.

Art. 29. Elementary Rules of Integration.

The first four rules of integration will be demonstrated in full.

(1) By I., $d(u + v - w) = du + dv - dw;$

hence $$\int d(u + v - w) = \int (du + dv - dw),$$

or $$u + v - w = \int (du + dv - dw).$$

But $$u + v - w = \int du + \int dv - \int dw;$$

therefore $$\int (du + dv - dw) = \int du + \int dv - \int dw.$$

Hence, *the integral of the algebraic sum of any number of differentials is equal to the algebraic sum of their integrals.*

(2) By II., $$d(av) = adv;$$

hence $$\int d(av) = \int a\,dv,$$

or $$av = \int a\,dv.$$

But $$av = a\int dv;$$

therefore $$\int a\,dv = a\int dv.$$

Hence, *the integral of the product of a constant and a differential is equal to the product of the constant and the integral of the differential.*

(3) By VII., $$dav^n = nav^{n-1}dv.$$

Then $$\int dav^n = \int nav^{n-1}dv;$$

$$av^n = \int nav^{n-1}dv$$

$$\int nav^{n-1}dv = \frac{nav^n}{n}.$$

Hence, *when a function consists of three factors,* — viz. *a constant factor, a variable factor with any constant exponent except* -1, *and a differential factor which is the differential of the variable without its exponent,* — *its integral is the product of the constant factor, by the variable factor with its exponent increased by* 1, *divided by the new exponent.*

(4) By VIII. *a*, $$da \log v = a\frac{dv}{v}.$$

Then $$\int da \log v = \int \frac{a\,dv}{v};$$

therefore $$\int \frac{a\,dv}{v} = a \log v.$$

Hence, *the integral of a fraction whose numerator is the product of a constant by the differential of the denominator, is equal to the product of the constant by the Naperian logarithm of the denominator.*

Art. 30. Constant of Integration.

By III., it is seen that the differential of a constant is zero; hence, constant terms disappear in differentiation. Therefore, in returning

from the differential to the integral, some constant must be added, which is called the *constant of integration.* The value of this arbitrary constant is determined in each case after integration by the data of the given problem, as will be shown hereafter. So, for the present, the undetermined constant will be omitted, but its addition after each integration will always be understood. Frequently, when a differential is integrated by different methods, the results may not appear to agree, but on inspection it will always be found that the integrals differ only by some constant.

If
$$y = f(x) + c, \tag{1}$$
$$\frac{dy}{dx} = f'(x), \tag{2}$$
$$dy = f'(x)dx, \tag{3}$$
$$\int dy = \int f'(x)dx, \tag{4}$$
$$y = \int f'(x)dx. \tag{5}$$

By equating the second members of (1) and (5),
$$\int f'(x)dx = f(x) + c. \tag{6}$$

This constant c which disappears in differentiating equation (1) is called the *constant of integration.* The integration in equation (5) gives only $f(x)$, so after every integration a constant c is added arbitrarily.

PROBLEMS.

FORMULAS 1–3.

1. $\int ax^3dx.$

By Formula 3, making $v = x$, and $n = 4$,
$$\int ax^3dx = \tfrac{1}{4}\int 4\,ax^3dx = \frac{ax^4}{4}.$$

2. $\int b(6\,ax^2 + 8\,bx^3)^{\frac{5}{3}}(2\,ax + 4\,bx^2)dx.$
$$d(6\,ax^2 + 8\,bx^3) = (12\,ax + 24\,bx^2)dx;$$

hence, if $(2\,ax + 4\,bx^2)dx$ be multiplied by 6, it will be the differential of $(6\,ax^2 + 8\,bx^3)^{\frac{5}{3}}$ without the parenthesis exponent. After dividing

the constant factor by 6 to preserve the same value, the integration may be effected by Formula 3, in which $v = 6ax^2 + 8bx^3$.

Therefore $$\int b(6ax^2 + 8bx^3)^{\frac{5}{3}}(2ax + 4bx^2)dx$$

$$= \int \frac{b}{6}(6ax^2 + 8bx^3)^{\frac{5}{3}}(12ax + 24bx^2)dx$$

$$= \frac{\frac{b}{6}(6ax^2 + 8bx^3)^{\frac{8}{3}}}{\frac{8}{3}} = \frac{b}{16}(6ax^2 + 8bx^3)^{\frac{8}{3}}.$$

3. $\displaystyle\int \frac{x\,dx}{\sqrt{a^2 + x^2}} = \int \tfrac{1}{2}(a^2 + x^2)^{-\frac{1}{2}} 2x\,dx = (a^2 + x^2)^{\frac{1}{2}}.$

4. $\displaystyle\int -\tfrac{2}{3}x^{-\frac{8}{5}}dx.$ *Ans.* $\frac{10}{9}x^{-\frac{3}{5}}.$

5. $\displaystyle\int (\tfrac{7}{2}ax^{\frac{5}{2}} - \tfrac{5}{2}bx^{\frac{3}{2}})dx.$ *Ans.* $ax^{\frac{7}{2}} - bx^{\frac{5}{2}}.$

6. $\displaystyle\int \frac{dx}{\sqrt{x}}.$ *Ans.* $2\sqrt{x}.$

7. $\displaystyle\int \left(\frac{12}{x^3} - \frac{5}{x^4}\right)dx.$ *Ans.* $-\dfrac{6}{x^2} + \dfrac{5}{3x^3}.$

8. $\displaystyle\int \frac{x^2dx}{(a^2 + x^3)^{\frac{1}{2}}}.$ *Ans.* $\frac{2}{3}(a^2 + x^3)^{\frac{1}{2}}.$

9. $\displaystyle\int (6x^4 + 2x^2 - 5)(3x^2 - 1)dx.$ *Ans.* $\dfrac{18x^7}{7} - \dfrac{17x^3}{3} + 5x.$

10. $\displaystyle\int (3ax^2 + 4bx^3)^{\frac{4}{3}}(2ax + 4bx^2)dx.$ *Ans.* $\frac{1}{7}(3ax^2 + 4bx^3)^{\frac{7}{3}}.$

11. $\displaystyle\int \frac{a\,dx}{x\sqrt{3bx + 4c^2x^2}}.$ *Ans.* $-\dfrac{2a\sqrt{3bx + 4c^2x^2}}{3bx}.$

12. $\displaystyle\int (b - x^2)^3x^{\frac{1}{2}}dx.$ *Ans.* $\frac{2}{3}b^3x^{\frac{3}{2}} - \frac{6}{7}b^2x^{\frac{7}{2}} + \frac{6}{11}bx^{\frac{11}{2}} - \frac{2}{15}x^{\frac{15}{2}}.$

Formulas 4–6.

13. $\displaystyle\int \frac{6x^2dx}{b + 2x^3}.$

$$\int \frac{6x^2dx}{b + 2x^3} = \int \frac{d(b + 2x^3)}{b + 2x^3} = \log(b + 2x^3), \text{ by Formula 4.}$$

14. $\int \frac{dx}{x-a}$. *Ans.* $\log(x-a)$.

15. $\int \frac{x^{n-1}dx}{a+bx^n}$. *Ans.* $\frac{1}{nb}\log(a+bx^n)$.

16. $\int \frac{\frac{1}{3}x\,dx}{x^2+\frac{3}{4}}$. *Ans.* $\log(x^2+\frac{3}{4})^{\frac{1}{6}}$.

17. $\int (\log x)^3\frac{dx}{x}$. *Ans.* $\frac{1}{4}(\log x)^4$.

18. $\int \frac{5x^3dx}{3x^4+7}$. *Ans.* $\log(3x^4+7)^{\frac{5}{12}}$.

19. $\int \frac{x^3dx}{x+1}$. *Ans.* $x-\frac{x^2}{2}+\frac{x^3}{3}-\log(x+1)$.

20. $\int ba^{2x}dx = \frac{b}{2\log a}\int a^{2x}\cdot\log a\cdot 2\,dx = \frac{ba^{2x}}{2\log a}$, by Formula 5.

21. $\int 3e^x dx$. *Ans.* $3e^x$.

22. $\int be^{ax}dx$. *Ans.* $\frac{b}{a}e^{ax}$.

23. $\int 3a^{x^2}x\log a\cdot dx$. *Ans.* $\frac{3}{2}a^{x^2}$.

24. $\int a^xe^xdx$. *Ans.* $\frac{a^xe^x}{1+\log a}$.

Formulas 7–14.

25. $\int \cos mx\,dx$.

$$\int \cos mx\,dx = \frac{1}{m}\int \cos mx\cdot d\,mx = \frac{1}{m}\sin mx, \text{ by Formula 7.}$$

26. $\int \sin^3(2x)\cos(2x)\,dx$.

$$\int \sin^3(2x)\cos(2x)\,dx = \tfrac{1}{2}\int \sin^3(2x)\cos(2x)\,2\,dx$$

$$= \tfrac{1}{2}\int \sin^3(2x)\,d\sin(2x) = \tfrac{1}{8}\sin^4(2x).$$

27. $\int \sec^2(x^3)\,x^2dx$. *Ans.* $\frac{1}{3}\tan x^3$.

28. $\int 5\sec(3x)\tan(3x)\,dx.$ *Ans.* $\frac{5}{3}\sec(3x).$

29. $\int\frac{\sin(3x)\,dx}{\cos^2(3x)}.$ *Ans.* $\frac{1}{3}\sec 3x.$

30. $\int e^{\cos x}\sin x\,dx.$ *Ans.* $-e^{\cos x}.$

31. $\int\frac{(1+\cos x)\,dx}{x+\sin x}.$ *Ans.* $\log[x+\sin x].$

32. $\int\tan x\,dx.$ *Ans.* $\log\sec x.$

33. $\int\sin\theta\sec^2\theta\,d\theta.$ *Ans.* $\sec\theta.$

34. $\int\cot x\,dx.$ *Ans.* $\log\sin x.$

35. $\int\frac{dx}{\sin x}=\int\frac{dx}{2\sin\frac{1}{2}x\cos\frac{1}{2}x}=\int\frac{\frac{1}{2}\sec^2(\frac{1}{2}x)dx}{\tan\frac{1}{2}x}=\log\tan\frac{1}{2}x.$

36. $\int\frac{dx}{\cos x}=\int\frac{dx}{\sin\left(\frac{\pi}{2}+x\right)}=\log\tan\left(\frac{\pi}{4}+\frac{1}{2}x\right).$

37. $\int\frac{dx}{\sin x\cos x}.$ *Ans.* $\log\tan x.$

38. $\int\frac{dx}{\sin^2 x\cos^2 x}.$ *Ans.* $\tan x-\cot x.$

FORMULAS 14–22.

39. $\int\frac{dx}{\sqrt{a^2-b^2x^2}}.$

$$\int\frac{dx}{\sqrt{a^2-b^2x^2}}=\int\frac{\frac{dx}{a}}{\sqrt{1-\frac{b^2x^2}{a^2}}}=\frac{1}{b}\int\frac{\frac{b}{a}dx}{\sqrt{1-\frac{b^2x^2}{a^2}}}=\frac{1}{b}\operatorname{arc\,sin}\frac{bx}{a}.$$

In order to integrate the preceding differential by Formula 15, it must be transformed into an equivalent differential having unity for the first term under the radical sign, and having for its numerator the differential of the square root of the second term under the radical sign.

40. $\int \frac{dx}{x\sqrt{b^2x^2 - a^2}}$.

$$\int \frac{dx}{x\sqrt{b^2x^2 - a^2}} = \int \frac{\frac{dx}{a}}{x\sqrt{\frac{b^2}{a^2}x^2 - 1}} = \frac{1}{a}\int \frac{\frac{b}{a}dx}{\frac{b}{a}x\sqrt{\frac{b^2}{a^2}x^2 - 1}}$$

$$= \frac{1}{a} \text{ arc sec } \frac{bx}{a}, \text{ by Formula 19.}$$

41. $\int \frac{dx}{\sqrt{2abx - b^2x^2}}$. *Ans.* $\frac{1}{b}$ arc vers $\frac{bx}{a}$.

42. $\int \frac{2x\,dx}{\sqrt{1 - x^4}}$. *Ans.* arc sin (x^2).

43. $\int - \frac{dx}{\sqrt{x - 4x^2}}$. *Ans.* arc cos $(2\sqrt{x})$.

44. $\int \frac{x\,dx}{1 + x^4}$. *Ans.* $\frac{1}{2}$ arc tan (x^2).

45. $\int \frac{8x^{-\frac{2}{3}}dx}{\sqrt{2x^{\frac{1}{3}} - 6x^{\frac{2}{3}}}}$. *Ans.* $4\sqrt{6}$ arc vers $(6x^{\frac{1}{3}})$.

46. $\int - \frac{2\,dx}{4 + x^2}$. *Ans.* arc cot $\frac{x}{2}$.

47. $\int \frac{dx}{2 - 2x + x^2}$. *Ans.* arc tan$(x - 1)$.

48. $\int \frac{dx}{x\sqrt{c^2x^2 - a^2b^2}}$. *Ans.* $\frac{1}{ab}$ arc sec $\frac{cx}{ab}$.

49. $\int - \frac{dx}{\sqrt{a^2x - b^2x^2}}$. *Ans.* $\frac{1}{b}$ arc covers $\frac{2b^2x}{a^2}$.

50. $\int \frac{dx}{1 + x + x^2} = \int \frac{dx}{\frac{3}{4} + (x + \frac{1}{2})^2} = \frac{2}{\sqrt{3}}\text{arc tan } (x + \frac{1}{2})\,\frac{2}{\sqrt{3}}$.

Art. 31. Integration of Trigonometric Differentials.

Trigonometric differentials, to which the previous formulas cannot be made applicable by algebraic reductions, may often be brought to known forms by trigonometric reductions.

In attempting to integrate any given differential, the object is always to transform it to a fundamental form whose integral is known. Hence, the processes of the Integral Calculus are transformations to effect reductions to fundamental formulas. In order to become proficient in these operations, it is necessary to have much practice in the solution of problems.

PROBLEMS.

1. $\int \cos^2 x\,dx.$

$$\int \cos^2 x\,dx = \int (\tfrac{1}{2} + \tfrac{1}{2}\cos 2x)\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x.$$

2. $\int \sin^2 x\,dx.$ *Ans.* $\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x.$

3. $\int \tan^3 x\,dx.$

$$\int \tan^3 x\,dx = \int (\sec^2 x - 1)\tan x\,dx = \tfrac{1}{2}\tan^2 x - \log \sec x.$$

4. $\int \tan^2 x\,dx.$ *Ans.* $\tan x - x.$

5. $\int \tan^4 x\,dx.$ *Ans.* $\tfrac{1}{3}\tan^3 x - \tan x + x.$

6. $\int \tan^5 x\,dx.$ *Ans.* $\tfrac{1}{4}\tan^4 x - \tfrac{1}{2}\tan^2 x + \log \sec x.$

7. $\int \sin^5 x\,dx.$

$$\int \sin^5 x\,dx = \int (1 - \cos^2 x)^2 \sin x\,dx = \int (-1 + 2\cos^2 x - \cos^4 x)\,d\cos x$$
$$= -\cos x + \tfrac{2}{3}\cos^3 x - \frac{\cos^5 x}{5}$$

8. $\int \cos^3 x\,dx.$ *Ans.* $\sin x - \tfrac{1}{3}\sin^3 x.$

9. $\int \cos^7 x\,dx.$ *Ans.* $\sin x - \sin^3 x + \tfrac{3}{5}\sin^5 x - \tfrac{1}{7}\sin^7 x.$

10. $\int \cot^4 x\,dx.$ *Ans.* $-\tfrac{1}{3}\cot^3 x + \cot x + x.$

11. $\int \cos^4 x\,dx = \int (\tfrac{1}{2} + \tfrac{1}{2}\cos 2x)^2 dx = \tfrac{1}{4}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{8}\int \cos^2 (2x)\,d(2x)$

$$= \tfrac{1}{4}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{8}[x + \tfrac{1}{4}\sin 4x] = \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x.$$

ART. 32. DEFINITE INTEGRALS.

It was shown in Art. 30, that an arbitrary constant must be added after each integration. Before the value of this constant is determined, the integral is said to be indefinite.

If, from the data of the given problem, the value of the integral is known for some particular value of the variable, the constant can be determined by substituting this value of the variable in the indefinite integral.

For example, let

$$\frac{dv}{dt} = a, \text{ or } dv = a\,dt.$$

Integrating and adding the constant C_1,

$$v = at + C_1. \tag{1}$$

Let $v = v_0$ when $t = 0$, and substituting these values in (1),

$$v_0 = 0 + C_1, \text{ or } C_1 = v_0,$$

and (1) becomes

$$v = at + v_0. \tag{2}$$

Thus the constant C, which was added arbitrarily after integration in (1), is determined if the value of the dependent variable v is known for a given value of the independent variable t.

Assume
$$v = \frac{ds}{dt},$$

then
$$\frac{ds}{dt} = at + v_0,$$

and
$$ds = atdt + v_0dt. \tag{3}$$

By integrating (3) and adding the constant C_2,

$$s = \frac{at^2}{2} + v_0t + C_2. \tag{4}$$

Let $s = s_0$ when $t = 0$, and substituting these values in (4),

$$s_0 = 0 + 0 + C^2, \text{ or } C_2 = s_0,$$

and (4) becomes

$$s = \frac{at^2}{2} + v_0t + s_0.$$

Thus the constant C_2, which was added arbitrarily after the integration in (3), is determined if the value of the dependent variable s is known for a given value of t.

Again, given $dy = (x^3 - 2x + 4)dx$, to find the value of the function y if $y = 0$ when $x = 2$.

$$\int dy = \int (x^3 - 2x + 4)dx,$$

and

$$y = \frac{x^4}{4} - x^2 + 4x + C. \tag{1}$$

Substituting $y = 0$ and $x = 2$ in (1),

$$0 = 4 - 4 + 8 + C, \text{ or } C = -8,$$

and (1) becomes

$$y = \frac{x^4}{4} - x^2 + 4x - 8.$$

The *definite integral* may also be obtained by *integration between limits*.

If, in any indefinite integral, two different values of the variable be substituted for the variable, and the result given by the second substitution be subtracted from the first result, the constant of integration is eliminated, and the integral is said to be *taken between limits.*

The definite integral of $f'(x)dx$ between the limits a and b is indicated thus:

$$\int_b^a f'(x)dx. \tag{1}$$

If $$y = f(x) + C,$$

and $$\frac{dy}{dx} = f'(x),$$

then $$y = \int f'(x)dx + C.$$

$$\int_b^a f'(x)dx + C = \Big[f(x) + C\Big]_b^a \tag{1}$$

$$= [f(a) + C] - [f(b) + C]$$

$$= f(a) - f(b).$$

Thus it will be seen that the *constant of integration*, C, disappears in integration between limits, and the result is a *definite integral*. In (1), a is the *superior* limit and b is the *inferior* limit of integration.

For example,

$$\int_2^3 3x^2dx = \Big[x^3\Big]_2^3$$

$$= 27 - 8 = 19.$$

It is assumed that the integral is continuous between the limits a and b. A function is said to be continuous between two values of the variable when it has a single finite value for every value of the variable between the given values, and changes gradually as the variable passes from the first value to the second. Evidently, the value of the integral up to the superior limit includes the value of the integral at the inferior limit. Hence, the difference between the values of the integral at two limits will be the value of the integral between those limits.

Assuming that the inferior limit is equal to b, and writing the integral in the two different ways;

$$\int f'(x)dx = f(x) + C, \tag{2}$$

and

$$\int_b^x f'(x)dx = f(x) - [f(x)]_b \tag{3}$$

If these two forms are taken to represent the same quantity,

$$f(x) + C \equiv f(x) - [f(x)]_b;\text{ whence } C \equiv [f(x)]_b. \tag{4}$$

Thus it will be seen that the upper limit is any final value of the increasing variable x, and that the lower limit may be assigned without defining the upper limit. Equation (4) shows that the constant C depends on the lower limit. Therefore, the integral in equation (2) is indefinite because a free choice is left with regard to the selection of both limits. The part $f(x)$ depends on the value of x selected for the superior limit, and the part C depends on the value taken for the inferior limit.

Art. 33. Geometric Illustration of Definite Integration.

The problem of finding the areas of plane curves was one of those that gave rise to the Integral Calculus, and this problem furnishes an illustration of the preceding article.

In Fig. 7, let MN represent any plane curve; it is required to find the area included between the curve, the X-axis and two ordinates.

Let (x, y) be the coördinates of the point P. If $RS = \Delta x$ be added to x, $SQ = y + \Delta y$. QC and PD are drawn parallel to the X-axis.

If A represents the area of the curve between two ordinates and the X-axis, ΔA = area $RPQS$.

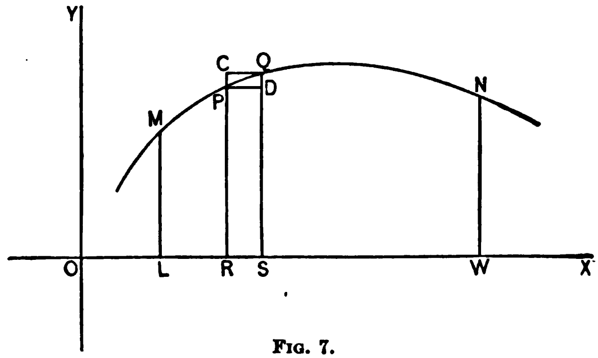

FIG. 7.

Then $$\frac{RCQS}{RPDS} = \frac{CR}{PR} = \frac{y + \Delta y}{y} = 1 + \frac{\Delta y}{y}.$$

Now, as Δx approaches zero, Δy also approaches zero;

and $$\text{limit } \frac{RCQS}{RPDS} = 1.$$

But the area $RPQS$ is intermediate between area $RCQS$ and area $RPDS$; hence

$$\text{limit } \frac{RPQS}{RPDS} = 1,$$

or $$\text{limit } \frac{\Delta A}{y \cdot \Delta x} = 1,$$

or $$\frac{dA}{y\,dx} = 1;$$

therefore $$dA = y\,dx,$$

and $$A = \int y\,dx + C. \qquad (1)$$

If the area between the ordinates NW and ML is required, the superior limit will be the abscissa OW, and the inferior limit will be

the abscissa OL. If these limits are respectively a and b, the area will be denoted by

$$A = \int_b^a y\,dx. \qquad (2)$$

Let the particular curve whose area is required be the common parabola, then $y = \sqrt{2\,px}$. Substituting this value of y in (1), gives

$$A = \int \sqrt{2\,px}\,dx + C = \sqrt{2\,p}\int x^{\frac{1}{2}}dx + C = \tfrac{2}{3}\sqrt{2\,p}x^{\frac{3}{2}} + C.$$

If the area is estimated from the origin, when $x = 0$, $A = 0$; hence, by the first method of Art. 32, $C = 0$, and, therefore, $A = \frac{2}{3}\sqrt{2\,p}x^{\frac{3}{2}}$.

If the area is required between two ordinates whose abscissas are a and b, by the second method of Art. 32,

$$\int_b^a \sqrt{2\,px}\,dx = \left[\tfrac{2}{3}\sqrt{2\,p}x^{\frac{3}{2}}\right]_b^a = \tfrac{2}{3}\sqrt{2\,p}\left[a^{\frac{3}{2}} - b^{\frac{3}{2}}\right].$$

Art. 34. Change of Limits.

Let
$$\int f'(x)\,dx = f(x);$$

then
$$\begin{aligned}\int_b^a f'(x)\,dx &= f(a) - f(b)\\ &= -\,[f(b) - f(a)]\\ &= -\int_a^b f'(x)\,dx.\end{aligned}$$

Hence, *the limits of integration may be interchanged by changing the sign of the integral.*

It may also be readily shown that

$$\int_c^b f'(x)\,dx = \int_c^a f'(x)\,dx + \int_a^b f'(x)\,dx.$$

If a new variable be substituted for the old variable in integration between limits, corresponding changes must be made in the limits of integration.

For example, $\int_1^4 x^n dx$ is required.

Suppose $x = z^2$; then when $x = 4$, $z = \pm 2$,
and when $x = 1$, $z = \pm 1$.

Therefore
$$\int_1^4 x^n dx = 2\int_{\pm 1}^{\pm 2} z^{2n+1} dz$$
$$= \frac{4^{n+1}-1}{n+1}.$$

PROBLEMS.

1. Find the particular integral of $dy = (x^3 - b^2 x)dx$, if $y = 0$ when $x = 2$.

$$Ans.\ y = \frac{x^4}{4} - \frac{b^2x^2}{2} + 2b^2 - 4.$$

2. Find the particular integral of $du = (1 + \frac{9}{4}ax)^{\frac{1}{2}}dx$, if $u = 0$ when $x = 0$.

$$Ans.\ u = \frac{8}{27a}(1 + \tfrac{9}{4}ax)^{\frac{3}{2}} - \frac{8}{27a}.$$

3. $\int_0^a (a^2x - x^3)\,dx = \left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{a^4}{4}.$

4. $2\int_0^{+0.2}(2 - 4e)\,de = 0.64.$

5. $\int_2^3 6\,x^2dx = 38.$

6. $\int_0^{\infty} \frac{dx}{a^2 + x^2} = \frac{\pi}{2a}.$

7. $\int_2^3 \frac{x\,dx}{1 + x^2} = \frac{\log 2}{2}.$

8. $\int_0^{\frac{\pi}{4}} \frac{\sin\theta d\theta}{\cos^2\theta} = \sqrt{2} - 1.$

9. $\int_0^{2r} \frac{\sqrt{2r}}{\sqrt{x}}dx = 4r.$

10. $\int_0^r \frac{r\,dx}{\sqrt{r^2 - x^2}} = \frac{1}{2}\pi r.$

11. $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^3 x\,dx = \frac{1}{12}.$

12. In $\int_0^a \frac{dx}{\sqrt{a^2 - x^2}}$, assume $z = \frac{x}{a}$. *Ans.* $\frac{\pi}{2}$.

13. In $\int_0^{\frac{\pi}{2}} \sin x \cos^2 x\,dx$, assume $\sin x = z$. *Ans.* $\frac{1}{3}$.

14. In $\int_0^1 \frac{x\,dx}{\sqrt{1 - x^2}}$, assume $y = 1 - x^2$. *Ans.* ± 1.

15. Find the area of the curve $y = x^2 + 6x$, between the abscissas $x = 6$ and $x = 0$. *Ans.* 180.

NOTE. Applications of the Integral Calculus in rectifying curves, determining areas, volumes, centre of mass, and moment of inertia, will be found in Chapters XIX., XX. and XXI.

ART. 35. VARIABLE OF INTEGRATION.

In the expression $\int y\,dx$, y must be a function of x, and x is called the *variable of integration*. If any other variable occurs after the integral sign it must be a function of x and its value in terms of x must be substituted before the integral is obtained.

If x and y are so related that

$$f'(x)dx = F'(y)dy$$

or

$$f'(x) = F'(y)\frac{dy}{dx},$$

then $\int F'(y)\,dy$ may be substituted for $\int f'(x)dx$.

Assume that

$$\int f'(x)\,dx = f(x)$$

and

$$\int F'(y)\,dy = F(y).$$

Now

$$\frac{d}{dx}f(x) = f'(x),$$

and

$$\frac{d}{dx}F(y) = \frac{d}{dy}F(y).\frac{dy}{dx} = F'(y)\frac{dy}{dx}.$$

The two functions therefore, having the same derivative, can differ only by a constant. These constants of integration may be so chosen that

$$f(x) = F(y).$$

In some problems the integration may be simplified by substituting for the given integral an equivalent integral in terms of a different variable.

ART. 36. MISCELLANEOUS PROBLEMS.

1. $\int (x^3 + 3x - 1)(x^2 + 1)dx.$

2. $\int (x+1)\sqrt{x^2+2x-1}\,dx.$

3. $\int \sqrt{\frac{1}{x}\sqrt{1-\sqrt{x}}}\,dx.$

4. $\int (1 + \epsilon^{3x})\epsilon^{3x}dx.$

5. $\int \frac{(2 + \log x)}{x}dx.$

6. $\int \frac{dx}{x(1 + \log x)}.$

7. $\displaystyle\int\frac{(x - x^3)dx}{x^4 - 2x^2 + 5}.$

8. $\displaystyle\int\frac{\log x\, dx}{x(1 - \log^2 x)}.$

9. $\displaystyle\int\frac{dx}{\sin 2x \cdot \log\tan x}.$

10. $\displaystyle\int x\, \epsilon^{x^2} dx.$

11. $\displaystyle\int\epsilon^{5x+1}dx.$

12. $\displaystyle\int\frac{\sin x}{\epsilon^{\cos x}}dx.$

13. $\displaystyle\int(\epsilon^x + \epsilon^{3x})^2 dx.$

14. $\displaystyle\int\frac{\epsilon^{x^2} + \sqrt{x}}{x^2}dx.$

15. $\displaystyle\int\sec(3x + 2)dx.$

16. $\displaystyle\int\cot(1 - x)dx.$

17. $\displaystyle\int(x\sec x^3)^2 dx.$

18. $\displaystyle\int\frac{\tan^2 2x \cdot dx}{\sin^2 2x}.$

19. $\displaystyle\int(\sin 2x + \cos 2x)^2 dx.$

20. $\displaystyle\int\frac{dx}{1 + \cos 2x}.$

21. $\displaystyle\int\frac{dx}{x^2 - 1}.$

22. $\displaystyle\int\frac{dx}{4 - 9x^2}.$

23. $\displaystyle\int\frac{dx}{2x^2 + 1}.$

24. $\displaystyle\int\frac{dx}{x^{\frac{2}{3}}(x^{\frac{2}{3}} + 1)}.$

25. $\displaystyle\int\frac{dx}{x(\log^2 x + 1)}.$

26. $\displaystyle\int\frac{dx}{\sqrt{2x - x^2}}.$

27. $\displaystyle\int\sin^3 ax \cdot \cos ax \cdot dx.$

28. $\displaystyle\int\frac{2x - 1}{2x + 3}dx.$

29. $\displaystyle\int\frac{x^2 + 1}{x - 1}dx.$

30. $\displaystyle\int(\epsilon^{\frac{x}{a}} + 1)^{\frac{1}{2}}\epsilon^{\frac{x}{a}}dx.$

31. $\displaystyle\int\epsilon^{x^2+4x+3}(x + 2)dx.$

32. $\displaystyle\int\frac{(a^x - b^x)^2}{a^x b^x}dx.$

33. $\displaystyle\int\operatorname{cosec} v.\, dv.$

34. $\displaystyle\int\frac{dy}{\cos^2 y}.$

35. $\displaystyle\int\frac{dx}{1 + \cos x}.$

36. $\displaystyle\int\frac{dx}{1 + \sin x}.$

37. $\displaystyle\int\frac{d\theta}{\sin^2 4\theta}.$

38. $\displaystyle\int(\sec\theta - 1)^2 d\theta.$

39. $\displaystyle\int x^3(2x^4 - 5)^3 dx.$

40. $\displaystyle\int\frac{\arctan x \cdot dx}{1 + x^2}.$

41. $\int \frac{dx}{x\sqrt{a^2x^2 - 1}}$.

42. $\int \frac{dx}{x^2 + 2x + 5}$.

43. $\int \epsilon^x \cos \epsilon^x dx$.

44. $\int \frac{x\,dx}{(1 + x^2)^2}$.

45. $\int \frac{x\,dx}{(2ax - x^2)^{\frac{3}{2}}}$.

46. $\int \frac{x\,dx}{\sqrt{1 + x}}$.

47. $\int x^2\sqrt{1 + x}\,dx$.

48. $\int x^3(x^2 + 1)^{-\frac{3}{2}} dx$.

49. $\int \frac{\sqrt{2ax - x^2}}{x^3} dx$.

50. $\int \frac{dx}{x^2\sqrt{x^2 - 1}}$.

CHAPTER VI.

SUCCESSIVE DIFFERENTIATION AND INTEGRATION.

ART. 37. SUCCESSIVE DERIVATIVES.

As the derivative of a function is, in general, a new function of the independent variable, it can be differentiated. The derivative of the first derivative is called the *second derivative.* Likewise, when the second derivative is a function of the independent variable, it may also be differentiated, giving the *third derivative;* and so on.

For example, if
$$y = ax^4;$$
$$\frac{dy}{dx} = 4\,ax^3,$$
$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 12\,ax^2,$$
$$\frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right] = 24\,ax, \text{ etc.}$$

The symbols for the successive derivatives are usually abbreviated as follows:
$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2},$$
$$\frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right] = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3},$$
$$\cdots \qquad \cdots \qquad \cdots$$
$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n}.$$

The successive derivatives are often called successive differential coefficients. As the first derivative is often denoted by $f'(x)$, the successive derivatives are often denoted by $f''(x)$, $f'''(x)$, etc.

If differentials are employed, successive differentials will follow instead of successive derivatives. The differential obtained immediately from the given function is the first differential; the differential of the first differential is the second differential; and so on. If the function be represented by y, the successive differentials will be denoted by dy, d^2y, d^3y, etc.

In successive differentiation it is customary to make the assumption that the differential of the independent variable is constant; *i.e.* the independent variable increases by equal increments, and hence is called an equicrescent variable. The independent variable will always be understood to be equicrescent unless the contrary is explicitly stated.

Art. 38. Successive Integration.

Two, three or n integrations must be performed in order to obtain the original function from which a second, third, or, in general, an nth derivative was derived.

For example, if $$\frac{d^3y}{dx^3} = 24\,ax. \tag{1}$$

Integrating (1), $$\frac{d^2y}{dx^2} = 12\,ax^2. \tag{2}$$

Integrating (2), $$\frac{dy}{dx} = 4\,ax^3. \tag{3}$$

Integrating (3), $$y = ax^4. \tag{4}$$

But an arbitrary constant should be added after each integration, as a constant term may have disappeared at each differentiation.

Then, in general, let $\frac{d^ny}{dx^n} = f(x)$; and denote the successive integrals of the function by $f_1(x)$, $f_2(x)$, $f_3(x)$, etc., and the constants of integration by C_1, C_2, C_3, etc.

Given $$\frac{d^ny}{dx^n} = f(x);$$

then $$\frac{d^{n-1}y}{dx^{n-1}} = f_1(x) + C_1,$$

$$\frac{d^{n-2}y}{dx^{n-2}} = f_2(x) + C_1 x + C_2,$$

$$\frac{d^{n-3}y}{dx^{n-3}} = f_3(x) + C_1 \frac{x^2}{2} + C_2 x + C_3,$$

$$\cdots \qquad \cdots$$

and finally,

$$y = f_n(x) + C_1 \frac{x^{n-1}}{1\cdot 2\cdot 3 \cdots (n-1)} + C_2 \frac{x^{n-2}}{1\cdot 2\cdot 3 \cdots (n-2)} \cdots + C_n.$$

PROBLEMS.

1. $y = ax^n.$

$$\frac{dy}{dx} = nax^{n-1};$$

$$\frac{d^2y}{dx^2} = n(n-1)\,ax^{n-2};$$

$$\frac{d^3y}{dx^3} = n(n-1)(n-2)\,ax^{n-3};$$

$$\cdots \qquad \cdots$$

$$\frac{d^ny}{dx^n} = \lfloor n \cdot a.$$

2. $y = \tan x.$

$$\frac{dy}{dx} = \sec^2 x;$$

$$\frac{d^2y}{dx^2} = 2\sec^2 x \cdot \tan x;$$

$$\frac{d^3y}{dx^3} = 6\sec^4 x - 4\sec^2 x;$$

$$\frac{d^4y}{dx^4} = 8\tan x \sec^2 x (3\sec^2 x - 1).$$

3. $y = ax^3 + bx^2.$ $\qquad \dfrac{d^4y}{dx^4} = 0.$

4. $y = \log(x+1).$ $\qquad \dfrac{d^4y}{dx^4} = -6(x+1)^{-4}.$

5. $y = a^x.$ $\qquad \dfrac{d^ny}{dx^n} = a^x(\log a)^n.$

6. $y = 6x^4 - 4x^3 - 5x^2.$ $\qquad \dfrac{d^4y}{dx^4} = 144.$

7. $y = \dfrac{1}{1+x^2}.$ $\qquad \dfrac{d^3y}{dx^3} = \dfrac{24x(1-x^2)}{(1+x^2)^4}.$

8. $y = x^4 \log(x^2).$ $\qquad \dfrac{d^5y}{dx^5} = \dfrac{48}{x}.$

9. $y = \tan^2 x + 8\log\cos x + 3x^2.$ $\qquad \dfrac{d^2y}{dx^2} = 6\tan^4 x.$

10. $y = xe^x$. $\qquad \dfrac{d^n y}{dx^n} = (x+n)e^x$.

11. $y = e^{x\cos a}\cos(x\sin a)$. $\qquad \dfrac{dy}{dx} = e^{x\cos a}\cos(x\sin a + a)$;

$$\frac{d^n y}{dx^n} = e^{x\cos a}\cos(x\sin a + na).$$

12. $\dfrac{d^3 y}{dx^3} = ax^2$. $\qquad y = \dfrac{ax^5}{60} + \dfrac{C_1 x^2}{2} + C_2 x + C_3$.

13. $\dfrac{d^3 y}{dx^3} = 2x^{-3}$. $\qquad y = \log x + C_1\dfrac{x^2}{2} + C_2 x + C_3$.

14. $\dfrac{d^4 y}{dx^4} = \cos x$. $\qquad y = \cos x + C_1\dfrac{x^3}{6} + C_2\dfrac{x^2}{2} + C_3 x + C_4$.

15. $\displaystyle\int_{a-c}^{a+c}\left(1 - \frac{a^2-c^2}{r^2}\right)dr = 0$. **16.** $\dfrac{\displaystyle\int_0^\pi y\,dx}{\displaystyle\int_0^\pi dx} = .6366\,a$, when $y = a\sin x$.

17. Find value of t from $a\dfrac{d^2x}{dt^2} = b(c-x)$.

Integrating once gives

$$a\frac{dx^2}{dt^2} = b(2cx - x^2);$$

hence
$$dt = \sqrt{\frac{a}{b}}\frac{dx}{\sqrt{2cx - x^2}}.$$

Therefore
$$t = \sqrt{\frac{a}{b}}\,\text{arc vers}\,\frac{x}{c}.$$

18. In the harmonic curve whose equation is $h = r_1\sin ml + r_2\cos ml$, find $\dfrac{d^2h}{dl^2}$; r_1, r_2 and m being constants.

Ans. $\dfrac{d^2h}{dl^2} = -m^2h$.

APPLICATIONS IN MECHANICS.

ART. 39. VELOCITY AND ACCELERATION OF MOTION.

The *mean velocity* of a moving body for a certain period, is equal to the distance passed over expressed in some unit of length, divided by the length of the period expressed in some unit of time. The velocity is *uniform* if equal distances are traversed in equal times; and the velocity is *variable* if unequal distances are traversed in equal times.

Let $s =$ distance, $v =$ velocity, and $t =$ time.

And let Δs denote the increment of distance passed over by the body in the increment of time Δt, while the velocity has increased to $v + \Delta v$.

The distance actually passed over, if the velocity is variable, lies between the distances it would have passed over if its velocities at the beginning and end of the period had been uniform; hence

$$v\,\Delta t < \Delta s < (v + \Delta v)\,\Delta t,$$

and

$$v < \frac{\Delta s}{\Delta t} < (v + \Delta v).$$

Now, as Δt approaches zero, Δv approaches zero, $(v + \Delta v)$ approaches v, and $\frac{\Delta s}{\Delta t}$ approaches $\frac{ds}{dt}$; and as the middle term is intermediate between the first and third terms, at the limit

$$\frac{ds}{dt} = v. \tag{1}$$

The *acceleration* at any instant is the rate at which the velocity is changing at that instant, and since the derivative of a function measures the rate at which its value is changing, if the acceleration is denoted by a,

$$\frac{dv}{dt} = a; \tag{2}$$

therefore

$$a = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}. \tag{3}$$

Equations (1), (2) and (3) are fundamental formulas.

ART. 40. UNIFORMLY ACCELERATED MOTION.

Motion is uniformly accelerated when the acceleration is constant. Denoting the acceleration, which may be positive or negative, by g; then from (3), Art. 39,

$$\int d\left(\frac{ds}{dt}\right) = \int g\,dt;$$

therefore
$$\frac{ds}{dt} = gt + C_1. \tag{4}$$

$$\int ds = \int gt\,dt + \int C_1 dt;$$

therefore
$$s = \tfrac{1}{2}gt^2 + C_1 t + C_2. \tag{5}$$

If, in (4), zero be substituted for t, C_1 will be equal to $\left(\frac{ds}{dt}\right)_{t=0}$; or C_1 will be the velocity at the beginning of the period, which may be represented by v_0.

If, in (5), zero be substituted for t, C_2 will be equal to the left member; or C_2 will be the distance already passed over at the beginning of the period, which may be denoted by s_0.

Making these substitutions in (4) and (5),

$$v = gt + v_0, \tag{6}$$

and
$$s = \tfrac{1}{2}gt^2 + v_0 t + s_0. \tag{7}$$

PROBLEMS.

1. If a body is dropped, what distance will it fall in 5 seconds, and what will be its velocity at the end of the fifth second, the acceleration of gravity being 32.2 feet per second?

In (6) and (7), $v_0 = 0$ and $s_0 = 0$; hence

$$v = gt,$$

and
$$s = \tfrac{1}{2}gt^2.$$

Substituting the values of g and t, gives

$v = 161$ feet per second, and $s = 402.5$ feet.

2. If a body is projected vertically upwards, to what height will it rise, and what will be the time of ascent?

In this case, the acceleration is negative, and $s_0 = 0$; hence equations (6) and (7) become

$$v = -gt + v_0, \tag{8}$$

and

$$s = -\tfrac{1}{2}gt^2 + v_0t. \tag{9}$$

When the body attains its greatest altitude, its velocity becomes zero. Therefore, if $v = 0$ in (8),

$$t = \frac{v_0}{g}, \tag{10}$$

which is the time during which the body rises.

Substituting $t = \dfrac{v_0}{g}$ in (9), gives

$$s = \frac{v_0^2}{2g}, \tag{11}$$

which is the height to which the body will rise.

3. A man is ascending in a balloon with a uniform velocity of 20 feet per second, when he drops a stone which reaches the ground in 4 seconds; find the height of the balloon. *Ans.* 176 feet.

4. A body is projected upwards with a velocity of 80 feet per second; in what time will it return to the place of starting? *Ans.* 5 seconds.

5. Two balls are dropped from a balloon, one of them 3 seconds before the other; how far will they be apart 5 seconds after the first one was dropped? *Ans.* 336 feet.

6. A body when first observed was falling at the rate of 40 feet per second, and struck the earth in 5 seconds; required the entire distance that the body fell.

ART. 41. DERIVATIVES OF THE PRODUCT OF TWO FUNCTIONS.

Let

$$y = uv, \tag{1}$$

u and v being functions of x; then, by IV.,

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}. \tag{2}$$

Differentiating (2) with respect to x, gives

$$\frac{d^2y}{dx^2}=v\frac{d^2u}{dx^2}+\frac{dv}{dx}\frac{du}{dx}+\frac{du}{dx}\frac{dv}{dx}+u\frac{d^2v}{dx^2}$$

$$=v\frac{d^2u}{dx^2}+2\frac{dv}{dx}\frac{du}{dx}+u\frac{d^2v}{dx^2}.$$

Similarly,

$$\frac{d^3y}{dx^3}=v\frac{d^3u}{dx^3}+\frac{dv}{dx}\frac{d^2u}{dx^2}+2\frac{dv}{dx}\frac{d^2u}{dx^2}+2\frac{d^2v}{dx^2}\frac{du}{dx}+\frac{d^2v}{dx^2}\frac{du}{dx}+\frac{d^3v}{dx^3}u$$

$$=v\frac{d^3u}{dx^3}+3\frac{dv}{dx}\frac{d^2u}{dx^2}+3\frac{d^2v}{dx^2}\frac{du}{dx}+\frac{d^3v}{dx^3}u.$$

By proceeding as above, the fourth derivative, and other successive derivatives, may be obtained, and it will be seen that the same law of the terms applies, the numerical coefficients being those of the Binomial Theorem; giving the general form,

$$\frac{d^ny}{dx^n}=v\frac{d^nu}{dx^n}+n\frac{dv}{dx}\frac{d^{n-1}u}{dx^{n-1}}+\frac{n(n-1)}{1\cdot 2}\frac{d^2v}{dx^2}\frac{d^{n-2}u}{dx^{n-2}}+\cdots$$

$$+n\frac{d^{n-1}v}{dx^{n-1}}\frac{du}{dx}+\frac{d^nv}{dx^n}u. \qquad (1)$$

This is known as Leibnitz's Formula.

That (1) is true for any nth derivative may be readily proved by mathematical induction.

Differentiating (1), and arranging the terms, gives

$$\frac{d^{n+1}y}{dx^{n+1}}=v\frac{d^{n+1}u}{dx^{n+1}}+(n+1)\frac{dv}{dx}\frac{d^nu}{dx^n}+\frac{n(n+1)}{1\cdot 2}\frac{d^2v}{dx^2}\frac{d^{n-1}u}{dx^{n-1}}+\cdots$$

$$+(n+1)\frac{d^nv}{dx^n}\frac{du}{dx}+\frac{d^{n+1}v}{dx^{n+1}}u. \qquad (2)$$

If the law of the terms expressed by (1) is true for n, it then appears from (2) that the formula is true when n is changed into $n+1$. But (1) has been shown to be true when n is 1, 2 or 3, then the formula must be true when n is 4, 5, 6, or any positive integer.

PROBLEMS.

Find the derivatives in the following examples, by the aid of Leibnitz's Theorem:

1. $y = x^3 \log x.$ $\qquad \dfrac{d^4y}{dx^4} = \dfrac{1 \cdot 2 \cdot 3}{x}.$

2. $y = e^{ax}z.$ $\qquad \dfrac{d^ny}{dx^n} = e^{ax}\left(a^nz + na^{n-1}\dfrac{dz}{dx} + \dfrac{n(n-1)}{1 \cdot 2}a^{n-2}\dfrac{d^2z}{dx^2}\cdots\right).$

3. $y = x^2a^x.$ $\qquad \dfrac{d^ny}{dx^n} = a^x(\log a)^{n-2}[(x \log a + n)^2 - n].$

CHAPTER VII.

FUNCTIONS OF TWO OR MORE VARIABLES. IMPLICIT FUNCTIONS. CHANGE OF THE INDEPENDENT VARIABLE.

ART. 42. PARTIAL DIFFERENTIATION.

If z be a function of two independent variables x and y, it may be expressed thus:

$$z = f(x, y). \tag{1}$$

In (1), z may be changed by changing either x or y.

For example, in the equation of a plane,

$$z = ax + by + c, \tag{2}$$

x and y are two independent variables, of which z is a function. In (2) a value may be given to either coördinate x or y without any reference to the other; so if either x or y receives an increment, z will take a corresponding increment. Then z may be differentiated with respect to x and y separately.

If (2) be differentiated, supposing x to vary and y to remain constant, the derivative is written

$$\frac{\partial z}{\partial x} = a. \tag{3}$$

If (2) be differentiated, supposing y to vary and x to remain constant, the derivative is written

$$\frac{\partial z}{\partial y} = b. \tag{4}$$

These derivatives are called *partial derivatives.*

According to the differential notation, equations (3) and (4) may be transformed into

$$\frac{\partial z}{\partial x} dx = a\,dx, \quad \text{and} \quad \frac{\partial z}{\partial y} dy = b\,dy,$$

and these expressions are called *partial differentials.*

Therefore, a partial differential of a function of several variables is a differential obtained on the hypothesis that only one of the variables changes.

A total differential of a function of several variables is a differential obtained on the hypothesis that all of the variables change.

To distinguish between the partial differentials of a function, the following notation is adopted: $\frac{\Delta z}{\Delta x}\Delta x$ and $\frac{\Delta z}{\Delta y}\Delta y$ will represent partial increments, and $\frac{\partial z}{\partial x}dx$ and $\frac{\partial z}{\partial y}dy$ partial differentials of z, with respect to x and y, respectively.

$D_x z$ and $\left(\frac{dz}{dx}\right)$ have been used to represent total derivatives of z with respect to x.

The general equation of a surface as given in Analytical Geometry of three dimensions is

$$z = f(x, y),$$

in which x and y are independent variables.

If the surface be cut by a plane parallel to the XZ plane, the equation of the curve of intersection will contain the variables x and z only, and the slope of the curve will be expressed by $\frac{\partial z}{\partial x}$.

Likewise, the equation of a section of the surface parallel to the YZ plane will contain the variables y and z only, and its slope will be $\frac{\partial z}{\partial y}$.

ART. 43. TOTAL DIFFERENTIAL OF A FUNCTION OF TWO OR MORE INDEPENDENT VARIABLES.

Let
$$z = f(x, y).$$

Let x and y be given successive increments Δx and Δy, and represent the corresponding total increment of the function by Δz.

Let
$$z' = f(x + \Delta x, y);$$

then
$$\frac{\Delta z}{\Delta x}\Delta x = f(x + \Delta x, y) - f(x, y), \tag{1}$$

$$\frac{\Delta z'}{\Delta y}\Delta y = f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y), \tag{2}$$

and $$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y). \tag{3}$$

Adding (1) and (2), and placing the first member of the resultant equation equal to the first member of (3), gives

$$\Delta z = \frac{\Delta z}{\Delta x}\Delta x + \frac{\Delta z'}{\Delta y}\Delta y. \tag{4}$$

Now, if Δx and Δy approach zero, limit $\Delta z' =$ limit Δz,

therefore, $$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Hence, the total differential of a function of two variables is equal to the sum of its partial differentials.

Similarly, the total differential of a function of any number of independent variables may be found to be equal to the sum of its partial differentials.

PROBLEMS.

1. $z = ax^3y^2$.

$$\frac{\partial z}{\partial x}dx = 3\,ax^2y^2dx;\ \frac{\partial z}{\partial y}dy = 2\,ax^3y\,dy.$$

therefore $$dz = 3\,ax^2y^2dx + 2\,ax^3y\,dy;$$

2. $z = x^y$. $\qquad dz = yx^{y-1}dx + x^y \log x\,dy.$

3. $z = \arctan\dfrac{y}{x}$. $\qquad dz = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$.

4. $z = \sin(xy)$. $\qquad dz = \cos(xy)[y\,dx + x\,dy]$.

5. $z = y^{\sin x}$. $\qquad dz = y^{\sin x}\log y \cos x\,dx + \dfrac{\sin x}{y^{\operatorname{covers} x}}dy.$

6. $u = x^{yz}$. $\qquad du = x^{yz-1}(yz\,dx + zx\log x\,dy + xy\log x\,dz).$

ART. 44. TOTAL DERIVATIVE OF u WITH RESPECT TO x WHEN $u = f(x, y, z)$, $y = \phi(x)$, and $z = \phi_1(x)$.

By Art. 43,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz;$$

therefore $$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx}. \tag{1}$$

COR. 1. If $u = f(x, y)$, and $y = \phi(x)$,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy;$$

therefore
$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}. \tag{2}$$

COR. 2. If $u = f(y, z)$, $y = \phi(x)$, and $z = \phi_1(x)$,

$$du = \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz;$$

therefore
$$\frac{du}{dx} = \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx}. \tag{3}$$

COR. 3. If $u = f(y)$, and $y = \phi(x)$,

$$du = \frac{du}{dy} dy;$$

therefore
$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}. \tag{4}$$

In the proposition, u is directly a function of x and also indirectly a function of x through y and z.

In Cor. 1, u is directly a function of x and indirectly a function of x through y.

In Cor. 2, u is indirectly a function of x through y and z.

In Cor. 3, u is indirectly a function of x through y.

PROBLEMS.

1. $u = e^{ax}(y - z)$, $y = a \sin x$, and $z = \cos x$.

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx}.$$

$$\frac{\partial u}{\partial x} = ae^{ax}(y - z),\ \frac{\partial u}{\partial y} = e^{ax},\ \frac{\partial u}{\partial z} = -e^{ax},$$

$$\frac{dy}{dx} = a\cos x,\ \frac{dz}{dx} = -\sin x;$$

therefore $$\frac{du}{dx} = ae^{ax}(y-z) + ae^{ax}\cos x + e^{ax}\sin x$$
$$= e^{ax}(a^2\sin x - a\cos x + a\cos x + \sin x)$$
$$= e^{ax}(a^2+1)\sin x.$$

2. $u = \arctan(xy)$, and $y = e^x$ $\qquad \dfrac{du}{dx} = \dfrac{e^x(1+x)}{1+x^2e^{2x}}.$

3. $u = yz$, $y = e^x$, and $z = x^4 - 4x^3 + 12x^2 - 24x + 24.$ $\qquad \dfrac{du}{dx} = e^x x^4.$

4. $u = \log(r^2 - y^2)$, and $y = r\sin\theta.$ $\qquad \dfrac{du}{d\theta} = -2\tan\theta.$

5. $u = \dfrac{e^{ax}(y-z)}{a^2+1}$, $y = a\sin x$, and $z = \cos x.$ $\qquad \dfrac{du}{dx} = e^{ax}\sin x.$

Art. 45. Successive Partial Derivatives of Two or More Variables.

If $u = f(x, y)$, then $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ are, in general, functions of both x and y, and may be differentiated with respect to either independent variable, giving *second partial derivatives.*

The partial derivative of $\dfrac{\partial u}{\partial x}$ with respect to x is $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial^2 u}{\partial x^2}.$

The partial derivative of $\dfrac{\partial u}{\partial y}$ with respect to y is $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial^2 u}{\partial y^2}.$

The partial derivative of $\dfrac{\partial u}{\partial x}$ with respect to y is $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial^2 u}{\partial y\,\partial x}.$

The partial derivative of $\dfrac{\partial u}{\partial y}$ with respect to x is $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial^2 u}{\partial x\,\partial y}.$

Likewise, $\dfrac{\partial^3 u}{\partial y^2 \partial x}$ is a *third partial derivative*, obtained by three successive differentiations; first, with respect to x regarding y as constant, and then twice with respect to y regarding x as constant.

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x\,\partial y}\right) = \frac{\partial^3 u}{\partial y\,\partial x\,\partial y},\quad \frac{\partial}{\partial x}\left(\frac{\partial^3 u}{\partial x\,\partial y^2}\right) = \frac{\partial^4 u}{\partial x^2 \partial y^2},$$

and similarly with all other partial derivatives.

ART. 46. IF $u = f(x, y)$, TO PROVE THAT $\dfrac{\partial^2 u}{\partial y\,\partial x} = \dfrac{\partial^2 u}{\partial x\,\partial y}$.

On the supposition that x alone changes in $f(x, y)$,

$$\frac{\Delta u}{\Delta x} = \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}. \tag{1}$$

Now, supposing y alone to change in (1),

$$\frac{\Delta}{\Delta y}\left(\frac{\Delta u}{\Delta x}\right) = \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) - f(x+\Delta x, y) + f(x, y)}{\Delta y \cdot \Delta x}. \tag{2}$$

On the supposition that y alone changes in $f(x, y)$,

$$\frac{\Delta u}{\Delta y} = \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}. \tag{3}$$

Now, supposing x alone to change in (3),

$$\frac{\Delta}{\Delta x}\left(\frac{\Delta u}{\Delta y}\right) = \frac{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) - f(x, y+\Delta y) + f(x, y)}{\Delta x \cdot \Delta y}. \tag{4}$$

Equating (2) and (4), $\dfrac{\Delta}{\Delta y}\left(\dfrac{\Delta u}{\Delta x}\right) = \dfrac{\Delta}{\Delta x}\left(\dfrac{\Delta u}{\Delta y}\right)$.

Hence at the limits, $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$.

In the same manner it may be proved that

$$\frac{\partial^3 u}{\partial x^2\,\partial y} = \frac{\partial^3 u}{\partial y\,\partial x^2} = \frac{\partial^3 u}{\partial x\,\partial y\,\partial x}.$$

This principle may be extended to any number of differentiations, and to functions of three or more variables.

ART. 47. IMPLICIT FUNCTIONS.

When in $f(x, y) = 0$, y can be expressed as an explicit function of x, the derivatives may be found by the methods already given. In this article a useful formula is established for obtaining the first derivative of an implicit function.

Let $$u = f(x, y) = 0. \tag{1}$$

Then by Art. 44, Cor. 1,

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}. \tag{2}$$

But $u = 0$, and therefore its total derivative equals 0; hence

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Solving (3) for $\frac{dy}{dx}$, gives $$\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}. \tag{4}$$

For example take $x^2 + 2yx + r^2 = 0$.

Then $$u = x^2 + 2yx + r^2.$$

$$\frac{\partial u}{\partial x} = 2x + 2y, \quad \frac{\partial u}{\partial y} = 2x.$$

Therefore by (4), $\frac{dy}{dx} = -\frac{2x + 2y}{2x} = -\frac{x + y}{x}$.

However, when an implicit relation between x and y is given from which y cannot readily be expressed as an explicit function of x, it is not necessary to resort to the method just given. But $f(x, y) = 0$ may be immediately differentiated with respect to x, treating y as a function of x, giving what is called the first derived equation, from which $\frac{dy}{dx}$ can be obtained as a function of x and y. For instance, given:

$$x^3 - 3axy + y^3 = 0.$$

The first derived equation will be

$$3x^2 - 3ay - 3ax\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0. \tag{1}$$

Solving (1) for $\frac{dy}{dx}$, $$\frac{dy}{dx} = \frac{x^2 - ay}{ax - y^2}.$$

PROBLEMS.

1. $u = \cos(x + y)$; verify $\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}$.

2. $u = \frac{y^2 + x^2}{y^2 - x^2}$; verify $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.

3. $u = \text{arc}\tan\left(\frac{y}{x}\right)$; verify $\frac{\partial^3 u}{\partial y^2\,\partial x} = \frac{\partial^3 u}{\partial x\,\partial y^2}$.

4. $u = 6\, e^x y^2 z + 3\, e^y x^2 z^2 + 2\, e^z x^3 y - xyz$;

$$\frac{\partial^4 u}{\partial x^2\, \partial y\, \partial z} = 12\,(e^x y + e^y z + e^z x).$$

5. $u = \sin(ax^n + by^n)$; verify $\dfrac{\partial^4 u}{\partial x^2\, \partial y^2} = \dfrac{\partial^4 u}{\partial y^2\, \partial x^2}$.

6. $y^2 - 2\, mxy + x^2 - a = 0$. $\dfrac{dy}{dx} = \dfrac{my - x}{y - mx}$, and $\dfrac{d^2y}{dx^2} = \dfrac{a(m^2-1)}{(y - mx)^3}$.

7. $y^3 - 3\, y + x = 0$. $\dfrac{dy}{dx} = \dfrac{1}{3\,(1 - y^2)}$.

8. $\left(\dfrac{x}{a}\right)^m + \left(\dfrac{y}{b}\right)^m = 1$. $\dfrac{dy}{dx} = -\left(\dfrac{b}{a}\right)^m \left(\dfrac{x}{y}\right)^{m-1}$.

9. $x^3 + 3\, axy + y^3 = 0$. $\dfrac{d^2y}{dx^2} = \dfrac{2\, a^3 xy}{(ax + y^2)^3}$.

Art. 48. Integration of Functions of Two or More Variables.

Since integration is the inverse of differentiation, a partial derivative is integrated by reversing the process of differentiation.

For example, the integral of $\dfrac{\partial^2 u}{\partial x^2} = f(x, y)$ is found by integrating twice with respect to x, regarding y as constant; but as y is regarded as a constant in this integration, it must be noticed that the constant of integration is an arbitrary function of y.

Again, let it be required to integrate $\dfrac{\partial^2 u}{\partial y\, \partial x} = f(x, y)$.

This may be expressed

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = f(x, y). \qquad (1)$$

Evidently, in the second differentiation, $\dfrac{\partial u}{\partial x}$ was differentiated with reference to y regarding x as constant; therefore

$$\frac{\partial u}{\partial x} = \int f(x, y)\, dy. \qquad (2)$$

In (2), u is evidently such a function that its derivative with respect to x is $\int f(x, y)\, dy$;

therefore,
$$u = \int \left[\int f(x, y)\, dy \right] dx,$$

or
$$u = \int \int f(x, y)\, dy\, dx.$$

In Art. 46, it was proved that the values of the partial derivatives are independent of the order in which the variables are supposed to change, hence the order of integration is also immaterial.

Similarly, if
$$\frac{\partial^3 u}{\partial y\, \partial x\, \partial y} = f(x, y),$$

then
$$u = \int \int \int f(x, y)\, dy\, dx\, dy;$$

and if
$$\frac{\partial^4 u}{\partial x^2 \partial y\, \partial z} = f(x, y, z),$$

then
$$u = \int \int \int \int f(x, y, z)\, dx^2 dy\, dz.$$

PROBLEMS.

1. $\displaystyle\int_{\frac{b}{2}}^{b} \int_0^{\frac{r}{b}} r\, dr\, d\theta = \frac{7\, b^2}{24}.$

2. $\displaystyle\int_1^2 \int_0^x \int_y^x xyz\, dx\, dy\, dz = \frac{21}{16}.$

Art. 49. Integration of Total Differentials of the First Order.

If u be a function of x and y, by Art. 43,
$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy. \tag{1}$$

And from Art. 46,
$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right). \tag{2}$$

Therefore, if a total differential of a function of x and y is given of the form
$$du = Pdx + Qdy, \tag{3}$$

then
$$P = \frac{\partial u}{\partial x}, \text{ and } Q = \frac{\partial u}{\partial y},$$

Hence from (2)
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \tag{4}$$

which is the condition that must be satisfied to make (3) an exact differential. This condition is called Euler's Criterion of Integrability. When (4) is satisfied, (3) is an exact differential of a function of x and y. Then the function u is obtained by integrating either term of (3); thus

$$u = \int P dx + f(y). \tag{5}$$

In (5), the integration is with respect to x, hence the constant of integration is an arbitrary function of the variable which is treated as a constant, and $f(y)$ must be determined so as to make $\frac{\partial u}{\partial y} = Q$.

For example, let $\quad du = 2xy^3dx + 3x^2y^2dy.$

Here $\quad P = 2xy^3$, and $Q = 3x^2y^2$.

Hence, $\quad \frac{\partial P}{\partial y} = 6xy^2$, and $\frac{\partial Q}{\partial x} = 6xy^2$.

Therefore (4) is satisfied, and (5) gives

$$u = \int 2xy^3dx = x^2y^3 + f(y) = x^2y^3 + c.$$

PROBLEMS.

1. $du = y\,dx + x\,dy.$ $\qquad u = xy + c.$
2. $du = 4x^3y^3dx + 3x^4y^2dy.$ $\qquad u = x^4y^3 + c.$
3. $du = \frac{dx}{y} + \left(2y - \frac{x}{y^2}\right)dy.$ $\qquad u = \frac{x}{y} + y^2 + c.$

ART. 50. CHANGE OF THE INDEPENDENT VARIABLE.

Hitherto, the derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, etc., have been obtained on the supposition that x was the independent variable and y the function, but it is sometimes advantageous to change the function into another one in which y is made the independent variable and x the function. And occasionally it is desirable to make a new variable, of which both x and y are functions, the independent variable.

(*a*) To express $\frac{dy}{dx}$ in terms of $\frac{dx}{dy}$.

If y is a function of x, then x may be regarded as a function of y, and y may be treated as the independent variable. Evidently

$$\frac{\Delta y}{\Delta x}\times\frac{\Delta x}{\Delta y}=1;$$

and as Δy approaches zero, Δx approaches zero, and at the limit,

$$\frac{dy}{dx}\times\frac{dx}{dy}=1;$$

therefore
$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}. \tag{1}$$

(*b*) To express $\frac{d^2y}{dx^2}$ in terms of $\frac{dx}{dy}$ and $\frac{d^2x}{dy^2}$, also to express $\frac{d^3y}{dx^3}$ in terms of $\frac{dx}{dy}$, $\frac{d^2x}{dy^2}$, and $\frac{d^3x}{dy^3}$.

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right).$$

From (1),
$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left\{\frac{1}{\frac{dx}{dy}}\right\}.$$

By Art. 44, Cor. 3,

$$\frac{d}{dx}\left\{\frac{1}{\frac{dx}{dy}}\right\}=\frac{d}{dy}\left\{\frac{1}{\frac{dx}{dy}}\right\}\cdot\frac{dy}{dx};$$

therefore
$$\frac{d^2y}{dx^2}=\frac{d}{dy}\left\{\frac{1}{\frac{dx}{dy}}\right\}\cdot\frac{dy}{dx}$$

$$=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2}\cdot\frac{dy}{dx}=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}. \tag{2}$$

Similarly, $$\frac{d^3y}{dx^3} = -\frac{d}{dx}\left[\frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3}\right] = -\frac{d}{dy}\left[\frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3}\right]\frac{dy}{dx}$$

$$= -\left[\frac{\left(\dfrac{dx}{dy}\right)^3\dfrac{d^3x}{dy^3} - 3\left(\dfrac{dx}{dy}\right)^2\left(\dfrac{d^2x}{dy^2}\right)^2}{\left(\dfrac{dx}{dy}\right)^6}\right]\frac{dy}{dx}$$

$$= -\frac{\left(\dfrac{dx}{dy}\right)\dfrac{d^3x}{dy^3} - 3\left(\dfrac{d^2x}{dy^2}\right)^2}{\left(\dfrac{dx}{dy}\right)^5}. \tag{3}$$

In equations (1), (2) and (3), the independent variable is changed from x to y.

(*c*) To express $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, etc., in terms of $\dfrac{dy}{dz}$, $\dfrac{d^2y}{dz^2}$, etc., when x is some given function of z.

By Art. 44, Cor. 3,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}; \tag{4}$$

therefore $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx}; \tag{5}$$

and $$\frac{d^3y}{dx^3} = \frac{d}{dz}\left(\frac{d^2y}{dx^2}\right)\frac{dz}{dx}. \tag{6}$$

In equations (4), (5) and (6), the independent variable is changed from x to z.

PROBLEMS.

1. Change the independent variable from x to y in

$$x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 - \frac{dy}{dx} = 0.$$

Substituting the values of $\dfrac{d^2y}{dx^2}$ and $\dfrac{dy}{dx}$ from (1) and (2),

$$-x\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}+\left\{\frac{1}{\frac{dx}{dy}}\right\}^3-\frac{1}{\frac{dx}{dy}}=0;$$

therefore
$$x\frac{d^2x}{dy^2}-1+\left(\frac{dx}{dy}\right)^2=0.$$

2. Change the independent variable from x to z in

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0,$$

when $x=\cos z$.

By (4),
$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx};$$

$$\frac{dx}{dz}=-\sin z,\ \text{hence}\ \frac{dz}{dx}=-\frac{1}{\sin z},$$

and
$$\frac{dy}{dx}=-\frac{1}{\sin z}\frac{dy}{dz}. \tag{7}$$

By (5),
$$\frac{d^2y}{dx^2}=\frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx};$$

$$\frac{d}{dz}\left(\frac{dy}{dx}\right)=\frac{d}{dz}\left(-\frac{1}{\sin z}\frac{dy}{dz}\right)$$

$$=\frac{\cos z}{\sin^2 z}\frac{dy}{dz}-\frac{1}{\sin z}\frac{d^2y}{dz^2}.$$

Therefore
$$\frac{d^2y}{dx^2}=-\left(\frac{\cos z}{\sin^2 z}\frac{dy}{dz}-\frac{1}{\sin z}\frac{d^2y}{dz^2}\right)\frac{1}{\sin z}$$

$$=-\left(\frac{\cos z}{\sin^3 z}\frac{dy}{dz}-\frac{1}{\sin^2 z}\frac{d^2y}{dz^2}\right). \tag{8}$$

Substituting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from (7) and (8) in the given example, gives

$$-(1-\cos^2 z)\left(\frac{\cos z}{\sin^3 z}\frac{dy}{dz}-\frac{1}{\sin^2 z}\frac{d^2y}{dz^2}\right)-\cos z\left(-\frac{1}{\sin z}\frac{dy}{dz}\right)=0.$$

Hence
$$\frac{d^2y}{dz^2}=0.$$

3. Given $y = f(x)$, and $x = F(t)$, to express $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in terms of $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$, $\frac{dy}{dt}$, and $\frac{d^2y}{dt^2}$.

By Art. 44, Cor. 3,
$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}. \tag{1}$$

Hence
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}. \tag{2}$$

Differentiating (1), and treating $\frac{dy}{dx}$ as a function of t through x, gives

$$\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\frac{dx^2}{dt^2} + \frac{dy}{dx}\frac{d^2x}{dt^2}.$$

Hence
$$\frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2} - \frac{dy}{dx}\frac{d^2x}{dt^2}}{\frac{dx^2}{dt^2}} = \frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}}{\frac{dx^3}{dt^3}}. \tag{3}$$

CHAPTER VIII.

DEVELOPMENT OF FUNCTIONS.

ART. 51. DEFINITION.

A Function is said to be developed when it is transformed into an equivalent series.

By the Binomial Theorem, constant powers of a binomial can be developed into series. For example,

$$(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$$

Some fractional functions may be developed by actual division. For example,

$$\frac{1}{1-3x} = 1 + 3x + 9x^2 + 27x^3 + \cdots$$

The Calculus method of development is a general method, including the developments just given and many others as special cases.

This is one of the most important applications of successive derivatives.

ART. 52. MACLAURIN'S THEOREM.

Maclaurin's Theorem is a theorem by which a function of a single variable may be developed into a series of terms arranged according to the ascending integral powers of that variable, with constant coefficients.

The function to be developed is

$$y = f(x).$$

Assume the development of the form

$$y = f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 \cdots \tag{1}$$

in which A, B, C, D, etc., are constants to be found by the method of Undetermined Coefficients.

Forming the successive derivatives of (1):

$$\frac{dy}{dx} = B + 2Cx + 3Dx^2 + 4Ex^3 + \cdots \tag{2}$$

$$\frac{d^2y}{dx^2} = 2C + 2\cdot 3Dx + 3\cdot 4Ex^2 + \cdots \tag{3}$$

$$\frac{d^3y}{dx^3} = 2\cdot 3D + 2\cdot 3\cdot 4Ex + \cdots \tag{4}$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

Since (1) and consequently (2), (3), etc., are assumed to be true for all values of x, they will be true when $x = 0$. Hence, making $x = 0$ in each of these equations, and representing what y becomes on this hypothesis by (y); what $\frac{dy}{dx}$ becomes by $\left(\frac{dy}{dx}\right)$; what $\frac{d^2y}{dx^2}$ becomes by $\left(\frac{d^2y}{dx^2}\right)$; and so on: there follows

from (1), $(y) = A$, or $A = (y)$;

from (2), $\left(\frac{dy}{dx}\right) = B$, or $B = \left(\frac{dy}{dx}\right)$;

from (3), $\left(\frac{d^2y}{dx^2}\right) = 2C$, or $C = \frac{1}{1\cdot 2}\left(\frac{d^2y}{dx^2}\right)$;

from (4), $\left(\frac{d^3y}{dx^3}\right) = 2\cdot 3D$, or $D = \frac{1}{1\cdot 2\cdot 3}\left(\frac{d^3y}{dx^3}\right)$;

$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$

Substituting these values of A, B, C, $\cdots$ in (1), gives

$$y = f(x) = (y) + \left(\frac{dy}{dx}\right)\frac{x}{1} + \left(\frac{d^2y}{dx^2}\right)\frac{x^2}{\underline{|2}} + \left(\frac{d^3y}{dx^3}\right)\frac{x^3}{\underline{|3}} + \cdots \tag{5}$$

If the function and its successive derivatives are expressed by

$$f(x),\ f'(x),\ f''(x),\ f'''(x),\ \text{etc.},$$

equation (5) may be written,

$$y=f(x)=f(0)+f'(0)\frac{x}{1}+f''(0)\frac{x^2}{\underline{|2}}+f'''(0)\frac{x^3}{\underline{|3}}+\cdots, \tag{6}$$

which is the formula of Maclaurin's Theorem.*

If in the attempted development of a function by Maclaurin's Theorem, the function or some one of its derivatives becomes infinite when $x=0$, the function cannot be developed by Maclaurin's Theorem. This is evident, because a finite function cannot be equal to a series containing infinite terms.

PROBLEMS.

1. To develop $y=(a+x)^n$.

Here

$$\begin{aligned}
f(x) &= (a+x)^n; & \text{hence } f(0) &= a^n.\\
f'(x) &= n(a+x)^{n-1}; & \text{“} \quad f'(0) &= na^{n-1}.\\
f''(x) &= n(n-1)(a+x)^{n-2}; & \text{“} \quad f''(0) &= n(n-1)a^{n-2}.\\
f'''(x) &= n(n-1)(n-2)(a+x)^{n-3}; & \text{“} \quad f'''(0) &= n(n-1)(n-2)a^{n-3}.\\
\cdots & \quad \cdots & \cdots & \quad \cdots
\end{aligned}$$

Substituting in (6), Art. 52,

$$y=(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{\underline{|2}}a^{n-2}x^2+\frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}x^3+\cdots,$$

which is the same development as that given by the Binomial Theorem.

2. To develop $y=\log(1+x)$.

Here,

$$\begin{aligned}
f(x) &= \log(1+x); & \text{hence } f(0) &= 0.\\
f'(x) &= \frac{m}{1+x}; & \text{“} \quad f'(0) &= m.\\
f''(x) &= -\frac{m}{(1+x)^2}; & \text{“} \quad f''(0) &= -m.\\
f'''(x) &= \frac{1\cdot 2\cdot m}{(1+x)^3}; & \text{“} \quad f'''(0) &= 1\cdot 2\,m.\\
\cdots & \quad \cdots & \cdots & \quad \cdots
\end{aligned}$$

* This theorem is commonly known as Maclaurin's, having been first published by him in 1742; but as it had been given in 1717 by Stirling, it should more properly bear the name of the latter.

Substituting in (6), Art. 52, gives

$$y = \log(1+x) = m\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right),$$

and if the logarithm is in the Naperian System,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

Thus the logarithmic series is found to be a special case under Maclaurin's Theorem.

3. To develop $y = \sin x$.

$$\begin{array}{llll} f(x) & = \sin x; & \text{hence } f(0) & = 0. \\ f'(x) & = \cos x; & \quad \text{"} \quad f'(0) & = 1. \\ f''(x) & = -\sin x; & \quad \text{"} \quad f''(0) & = 0. \\ f'''(x) & = -\cos x; & \quad \text{"} \quad f'''(0) & = -1. \\ \cdots & \cdots & \qquad \cdots & \cdots \end{array}$$

Therefore, $$y = \sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots.$$

In the successive derivatives of $\sin x$, the first four values are periodically repeating; *i.e.*, the fifth derivative equals the first, the sixth equals the second, etc.;

hence, in general, $$\frac{d^n(\sin x)}{dx^n} = \sin\left(x + n\frac{\pi}{2}\right).$$

Obtain by Maclaurin's Theorem the following developments:

4. $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} \cdots.$

The general formula for the successive derivatives of $\cos x$ is

$$\frac{d^n(\cos x)}{dx^n} = \cos\left(x + n\frac{\pi}{2}\right).$$

By the aid of the last two developments, natural sines and cosines may be computed.

For example, to find the sine of 45°.

By Art. 22, the circular measure of 45° is $\frac{\pi}{4}$. Substituting this value of x in the series of Prob. 3, gives

$$\sin 45° = \frac{\pi}{4} - \frac{1}{\lfloor 3}\left(\frac{\pi}{4}\right)^3 + \frac{1}{\lfloor 5}\left(\frac{\pi}{4}\right)^5 - \frac{1}{\lfloor 7}\left(\frac{\pi}{4}\right)^7 + \cdots$$
$$= .7071068.$$

5. $a^x = 1 + \log a \frac{x}{1} + \log^2 a \frac{x^2}{\lfloor 2} + \log^3 a \frac{x^3}{\lfloor 3} + \cdots.$

If $a = e$ and $x = 1$ in this series, the value of the Naperian base may be computed.

6. $\text{arc} \tan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$

The labor in finding the successive derivatives may sometimes be lessened by expanding the first derivative by some one of the algebraic methods, as follows:

$$f(x) = \tan^{-1} x.$$

$$f'(x) = \frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \cdots.$$

$$f''(x) = -2x + 4x^3 - 6x^5 + \cdots.$$

$$\cdots \qquad \cdots$$

By substituting $x = 1$ in the development of arc tan x,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

By Trigonometry, arc tan 1 = arc tan $\frac{1}{2}$ + arc tan $\frac{1}{3}$.

Hence $\frac{\pi}{4} = \left[\frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 - \cdots\right] + \left[\frac{1}{3} - \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 - \cdots\right].$

Therefore $\pi = 3.141592+.$

7. $\sqrt{1 + x^2} = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^8}{128} + \cdots.$

8. $e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{2 \cdot 4} - \frac{x^5}{3 \cdot 5} - \cdots.$

9. $e^x \sec x = 1 + x + x^2 + \frac{2x^3}{3} + \cdots.$

10. $(a^5 + a^4 x - x^5)^{\frac{1}{5}} = a + \frac{x}{5} - \frac{4}{5^2 a} \frac{x^2}{\lfloor 2} + \frac{4 \cdot 9}{5^3 a^2} \frac{x^3}{\lfloor 3} - \cdots.$

11. $\text{arc} \sin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3 x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots.$

It was by the use of this series that Sir Isaac Newton computed the value of π.

12. Develop $y = \log x$.

$$f(x) = \log x;\ \text{hence}\ f(0) = -\infty.$$
$$f'(x) = \frac{1}{x};\ \text{hence}\ f'(0) = \infty.$$
$$f''(x) = -\frac{1}{x^2};\ \text{hence}\ f''(0) = -\infty.$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

Substituting in Maclaurin's Formula, gives,

$$y = \log x = -\infty + \infty\frac{x}{1} - \infty\frac{x^2}{2} + \cdots.$$

In this example, $\log x$ equals a series of terms involving ∞, which makes the development indeterminate for all values of x.

Hence this function cannot be developed by Maclaurin's Theorem.

13. Develop $y = \cot x$.

14. If e be substituted for a in Ex. 5,

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \cdots. \tag{1}$$

Substituting $x\sqrt{-1}$ for x,

$$e^{x\sqrt{-1}} = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \cdots + \sqrt{-1}\left(x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \cdots\right)$$
$$= \cos x + \sqrt{-1}\sin x,\ \text{by Exs. 4 and 3.} \tag{2}$$

Substituting $-x\sqrt{-1}$ for x in (1), gives similarly

$$e^{-x\sqrt{-1}} = \cos x - \sqrt{-1}\sin x. \tag{3}$$

Combining (2) and (3), gives

$$\sin x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}},$$

and

$$\cos x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}.$$

These values of the sine and cosine are called their *exponential values*.

The real functions, $\frac{e^x - e^{-x}}{2}$ and $\frac{e^x + e^{-x}}{2}$, are called respectively the *hyperbolic sine* and *hyperbolic cosine* of x and are written $\sinh x$ and $\cosh x$.

ART. 53. TAYLOR'S THEOREM.

Taylor's Theorem is a theorem for developing a function of the sum of two variables into a series of terms arranged according to the ascending powers of one of the variables, with coefficients that are functions of the other variable.

Taylor's Theorem depends on the following principle which must first be established: The derivative of a function of the sum of x and y with reference to x regarding y as constant, is equal to the derivative of the function with reference to y regarding x as constant.

Let $$u = f(x+y).$$

Substituting $$z = x + y, \text{ gives}$$

$$u = f(z).$$

In the first case, $$\frac{\partial u}{\partial x} = \frac{df(z)}{dz}\frac{\partial z}{\partial x}.$$

$$= f'(z), \text{ since } \frac{\partial z}{\partial x} = 1.$$

In the second case, $$\frac{\partial u}{\partial y} = \frac{df(z)}{dz}\frac{\partial z}{\partial y}$$

$$= f'(z), \text{ since } \frac{\partial z}{\partial y} = 1.$$

Therefore $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}.$$

This principle may readily be shown to apply in any particular example; for instance,

let $$u = (x+y)^n,$$

then $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = n(x+y)^{n-1}.$$

ART. 54. DEMONSTRATION OF TAYLOR'S THEOREM.

Let $$u = f(x+y).$$

Assume the development to have the form

$$u = f(x+y) = A + By + Cy^2 + Dy^3 + \cdots, \tag{1}$$

in which $A, B, C, \cdots$ are independent of y, but are functions of x. It is now required to find values of $A, B, C, \cdots$ by the method of undetermined coefficients.

Differentiating (1), first with reference to x, regarding y as constant, then with reference to y, regarding x as constant,

$$\frac{\partial u}{\partial x}=\frac{dA}{dx}+\frac{dB}{dx}y+\frac{dC}{dx}y^2+\frac{dD}{dx}y^3+\cdots,$$

$$\frac{\partial u}{\partial y}=B+2\,Cy+3\,Dy^2+4\,Ey^3+\cdots.$$

But by Art. 53, $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}$; therefore,

$$\frac{dA}{dx}+\frac{dB}{dx}y+\frac{dC}{dx}y^2+\frac{dD}{dx}y^3=B+2\,Cy+3\,Dy^2+4\,Ey^3+\cdots. \qquad (2)$$

Making $y=0$ in (1), gives $A=f(x)$.

Since (2) is true for every value of y; equating the coefficients of like powers of y in the two members by the principle of Undetermined Coefficients,

$$\frac{dA}{dx}=B, \quad \text{hence } B=\frac{df(x)}{dx} \qquad =f'(x);$$

$$\frac{dB}{dx}=2\,C, \quad \text{hence } C=\frac{1}{2}\frac{d}{dx}(f'(x) \qquad =\frac{1}{\lfloor 2}f''(x);$$

$$\frac{dC}{dx}=3\,D, \quad \text{hence } D=\frac{1}{3}\frac{d}{dx}\left(\frac{1}{\lfloor 2}f''(x)\right)=\frac{1}{\lfloor 3}f'''(x); \text{ etc.}$$

Substituting these values of $A, B, C, \cdots$ in (1), gives

$$u=f(x+y)=f(x)+f'(x)\,y+f''(x)\frac{y^2}{\lfloor 2}+f'''(x)\frac{y^3}{\lfloor 3}+\cdots, \qquad (3)$$

which is Taylor's Theorem.

If $x=0$ be substituted in (3), it reduces to

$$f(y)=f(0)+f'(0)\,y+f''(0)\frac{y^2}{\lfloor 2}+f'''(0)\frac{y^3}{\lfloor 3}+\cdots,$$

which is Maclaurin's Theorem. So Maclaurin's Theorem may be considered as being but a special case of the more general one, Taylor's Theorem.*

PROBLEMS.

1. To develop $(x+y)^n$.

Substituting $y = 0$, and taking the successive derivatives,

$$\begin{aligned} f(x) &= x^n, \\ f'(x) &= nx^{n-1}, \\ f''(x) &= n(n-1)x^{n-2}, \\ f'''(x) &= n(n-1)(n-2)x^{n-3}. \\ \cdots & \quad \cdots \end{aligned}$$

Substituting these values in (3),

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{\underline{|2}}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^{n-3}y^3 + \cdots,$$

which is the Binomial Formula.

2. To develop $\sin(x+y)$.

$$\begin{aligned} f(x) &= \sin x, & f'(x) &= \cos x, \\ f''(x) &= -\sin x, & f'''(x) &= -\cos x. \\ & \cdots & & \cdots \end{aligned}$$

Therefore $\sin(x+y) = \sin x\left(1 - \frac{y^2}{\underline{|2}} + \frac{y^4}{\underline{|4}} - \cdots\right) + \cos x\left(y - \frac{y^3}{\underline{|3}} + \frac{y^5}{\underline{|5}} - \cdots\right)$

$$= \sin x \cos y + \cos x \sin y.$$

Obtain the following developments by Taylor's Theorem:

3. $a^{x+y} = a^x(1 + \log a \cdot y + \log^2 a\frac{y^2}{\underline{|2}} + \log^3 a\frac{y^3}{\underline{|3}} + \cdots.)$

4. $\log(x+y) = \log x + \frac{y}{x} - \frac{1}{2}\frac{y^2}{x^2} + \frac{1}{3}\frac{y^3}{x^3} - \cdots.$

5. $(x+y)^{\frac{1}{3}} = x^{\frac{1}{3}} + \frac{1}{3}x^{-\frac{2}{3}}y - \frac{1}{9}x^{-\frac{5}{3}}y^2 + \frac{5}{81}x^{-\frac{8}{3}}y^3 - \cdots.$

* Taylor's Theorem is named from its discoverer, Dr. Brook Taylor. It was first published in 1715, in a book by Dr. Taylor entitled *Methodus Incrementorum Directa et Inversa*.

6. $\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots.$

7. $\log \sec(x+y) = \log \sec x + \tan x \cdot y + \sec^2 x \cdot \frac{y^2}{2}$

$$+ \sec^2 x \cdot \tan x \cdot \frac{y^3}{3} \cdots.$$

Art. 55. Rigorous Proof of Taylor's Theorem.

In the demonstrations of Taylor's and Maclaurin's Theorems, it was assumed that the development would take place in a proposed form, and an infinite series was used without ascertaining that it was convergent. On account of these, as well as other objections, the method used is not altogether satisfactory. But, on the other hand, a rigorous investigation is necessarily complex and indirect. The proof which follows is one of the least difficult ones.

The following proposition must be first established:

If $\phi(x) = 0$, when $x = a$, and also when $x = b$, and if $\phi(x)$ and $\phi'(x)$ are finite and continuous between these values; then $\phi'(x)$ will vanish for some value of x between a and b.

The limit of $\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$, and hence $\frac{dy}{dx}$ will have the same sign as $\frac{\Delta y}{\Delta x}$ when Δx is taken small enough. If y increases as x increases, $\frac{\Delta y}{\Delta x}$ will be positive, and if y decreases as x increases, $\frac{\Delta y}{\Delta x}$ will be negative. So, if $\frac{dy}{dx}$ is always positive between the two given values of x, $\phi(x)$ would be constantly increasing, and if $\frac{dy}{dx}$ is always negative between the two values of x, $\phi(x)$ would be constantly decreasing; but neither supposition can be true, as $\phi(x)$ vanishes at the two given values for x.

Therefore, $\phi'(x)$ must change its sign between the two values, but a variable can only change its sign by passing through zero or infinity, and $\phi'(x)$ remains finite by hypothesis; hence, $\phi'(x)$ must pass through the value zero.

Let $f(x)$ and its successive derivatives be finite and continuous between $x = a$, and $x = a + h$.

Assume

$$\phi(x) = f(a+x) - f(a) - xf'(a) - \frac{x^2}{\lfloor 2}f''(a) \cdots - \frac{x^n}{\lfloor n}f^n(a) - \frac{x^{n+1}}{\lfloor n+1}R, \quad (1)$$

in which

$$R = \frac{\lfloor n+1}{h^{n+1}}\left[f(a+h) - f(a) - hf'(a) - \frac{h^2}{\lfloor 2}f''(a) \cdots - \frac{h^n}{\lfloor n}f^n(a)\right]. \quad (2)$$

In (2), it is to be observed that R is independent of x.

In (1), it is evident that $\phi(x) = 0$ when $x = 0$, and when $x = h$.

Hence, $\phi'(x)$ must be equal to zero for some small value of x between 0 and h. Represent this value by x_1.

Taking the derivative of (1) with respect to x,

$$\phi'(x) = f'(a+x) - f'(a) - xf''(a) - \frac{x^2}{\lfloor 2}f'''(a) \cdots - \frac{x^{n-1}}{\lfloor n-1}f^n(a) - \frac{x^n}{\lfloor n}R \quad (3)$$

$$= 0, \text{ when } x = x_1.$$

But (3) also vanishes when $x = 0$; hence there is some value of x between 0 and x_1, for which $\phi''(x) = 0$.

Continuing this process to $n+1$ differentiations,

$$\phi^{n+1}(x) = f^{n+1}(a+x) - R,$$

for some value of x between 0 and h, $\phi^{n+1}(x) = 0$; let this value of x be θh, where $\theta < 1$, therefore

$$f^{n+1}(a+\theta h) = R. \quad (4)$$

Equating the values of R in (2) and (4), and solving for $f(a+h)$,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2}f''(a) \cdots + \frac{h^n}{\lfloor n}f^n(a) + \frac{h^{n+1}}{\lfloor n+1}f^{n+1}(a+\theta h). \quad (5)$$

Now, since the only restriction imposed on a was that it must be finite, a may have any value; hence x may be substituted for a in (5), which gives

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2}f''(x) \cdots + \frac{h^n}{\lfloor n}f^n(x) + \frac{h^{n+1}}{\lfloor n+1}f^{n+1}(x+\theta h). \quad (6)$$

From (6), Taylor's Theorem follows whenever the function is such

that by sufficiently increasing n the last term can be made indefinitely small.*

ART. 56. REMAINDER IN TAYLOR'S AND MACLAURIN'S THEOREMS.

The last term of (6), Art. 55, $\frac{h^{n+1}}{\lfloor n+1}f^{n+1}(x+\theta h)$, is called the remainder after $n+1$ terms.

For example, let $f(x) = (1+x)^m$, then by (6), Art. 55,

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{\lfloor 2}x^2 + \cdots$$
$$+ \frac{x^{n+1}}{\lfloor n+1}[m(m-1)\cdots(m-n)(1+\theta x)^{m-n-1}].$$

In this development, $\frac{x^{n+1}}{\lfloor n+1}[m(m-1)\cdots(m-n)(1+\theta x)^{m-n-1}]$ is the remainder.

If x is less than 1, the last term can be made indefinitely small by sufficiently increasing m.

Hence, when $x<1$, $1+mx+\frac{m(m-1)}{\lfloor 2}x^2+\cdots$ is a convergent series.

If $x=0$ be substituted in the remainder in Taylor's Theorem, and then x be substituted for h, the remainder in Maclaurin's Theorem is obtained, which is

$$\frac{x^{n+1}}{\lfloor n+1}f^{n+1}(\theta x).$$

If this remainder, when n is taken sufficiently large, becomes indefinitely small, Maclaurin's Theorem gives a convergent series.

For example, let $f(x) = \sin x$.

Then $\sin(x) = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots \frac{x^{n+1}}{\lfloor n+1}\sin\left[\theta x + (n+1)\frac{\pi}{2}\right]$.

But whatever may be the value of x, it is evident that

$$\frac{x^{n+1}}{\lfloor n+1}\sin\left[\theta x + (n+1)\frac{\pi}{2}\right]$$

* The proof of Taylor's Theorem given in this article is due to Mr. Homersham Cox.

will have zero for its limit, hence this series is convergent for all values of x.

ART. 57. TAYLOR'S THEOREM FOR FUNCTIONS OF TWO OR MORE INDEPENDENT VARIABLES.

Let $f(x, y)$ be a function of two independent variables, and suppose $f(x+h, y+k)$ is to be expanded in ascending powers of h and k.

Regarding y as constant, and expanding as though x was the only variable,

$$f(x+h, y+k) = f(x, y+k) + h\frac{\partial}{\partial x}f(x, y+k) + \frac{h^2}{2}\frac{\partial^2}{\partial x^2}f(x, y+k) + \cdots. \quad (1)$$

Expanding $f(x, y+k)$, regarding x as constant, and y as the only variable,

$$f(x, y+k) = f(x, y) + k\frac{\partial}{\partial y}f(x, y) + \frac{k^2}{2}\frac{\partial^2}{\partial y^2}f(x, y) + \cdots. \quad (2)$$

Substituting this value of $f(x, y+k)$ from (2) in (1), gives

$$\begin{aligned} f(x+h, y+k) &= f(x, y) + h\frac{\partial}{\partial x}f(x, y) + k\frac{\partial}{\partial y}f(x, y) \\ &+ \frac{1}{\underline{|2}}\left[h^2\frac{\partial^2}{\partial x^2}f(x, y) + 2hk\frac{\partial^2}{\partial x\,\partial y}f(x, y) + k^2\frac{\partial^2}{\partial y^2}f(x, y)\right] \\ &+ \frac{1}{\underline{|3}}\left[h^3\frac{\partial^3}{\partial x^3}f(x, y) + 3h^2k\frac{\partial^3}{\partial x^2\partial y}f(x, y) + 3hk^2\frac{\partial^3}{\partial x\,\partial y^2}f(x, y)\right. \\ &\left. + k^3\frac{\partial^3}{\partial x\,\partial y^3}f(x, y)\right] + \cdots. \end{aligned}$$

Similarly,

$$\begin{aligned} &f(x+h, y+k, z+l) \\ &= f(x, y, z) + h\frac{\partial}{\partial x}f(x, y, z) + k\frac{\partial}{\partial y}f(x, y, z) + l\frac{\partial}{\partial z}f(x, y, z) \\ &+ \frac{1}{\underline{|2}}\left[h^2\frac{\partial^2}{dx^2}f(x, y, z) + 2hk\frac{\partial^2}{\partial x\,\partial y}f(x, y, z) + k^2\frac{\partial^2}{\partial y^2}f(x, y, x) + \cdots\right] \\ &+ \frac{1}{\underline{|3}}\left[h^3\frac{\partial^3}{dx^3}f(x, y, z) + 3h^2k\frac{\partial^3}{\partial x^2\partial y}f(x, y, z) + \cdots\right] + \cdots. \end{aligned}$$

And in like manner a function of any number of independent variables may be expanded.

CHAPTER IX.

EVALUATION OF INDETERMINATE FORMS.

ART. 58. INDETERMINATE FORMS.

A function of x is *indeterminate* when the substitution of a particular value for x gives rise to one of the following expressions:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ \infty - \infty,\ 0^0,\ \infty^0,\ 1^{\pm\infty}.$$

The *true value* of a function which becomes indeterminate is the value which the function approaches as its limit, as the independent variable approaches the particular value which makes the function indeterminate.

For example, to find the true value of $\dfrac{x^2 - a^2}{x - a}$ when $x = a$.

When $x = a$, this fraction assumes the form $\dfrac{0}{0}$.

If $a + h$ is substituted for x, the fraction becomes

$$\frac{(a + h)^2 - a^2}{a + h - a} = 2a + h.$$

Now if h approaches zero, the independent variable approaches the particular value a, and the function evidently approaches $2a$ as its true value.

Again, if both numerator and denominator of the fraction $\dfrac{x^2 - a^2}{x - a}$ are divided by $x - a$, the quotient is $x + a$, and now when $x = a$, the true value is found as before to be $2a$.

As another example,

$$\frac{a - \sqrt{a^2 - x^2}}{x^2} = \frac{0}{0},\ \text{when } x = 0.$$

By rationalizing the numerator,

$$\frac{a-\sqrt{a^2-x^2}}{x^2}=\frac{a^2-(a^2-x^2)}{x^2(a+\sqrt{a^2-x^2})}=\left[\frac{1}{a+\sqrt{a^2-x^2}}\right]_{x=0}=\frac{1}{2a}.$$

By algebraic and trigonometric transformations the true values of some indeterminate forms can be readily found, but the Differential Calculus furnishes a method of very general application.

Art. 59. Functions that take the Form $\frac{0}{0}$.

Let $f(x)$ and $\phi(x)$ be two functions, such that $f(x)=0$ and $\phi(x)=0$, when $x=a$;

then $$\frac{f(a)}{\phi(a)}=\frac{0}{0}.$$

Let x take an increment h; then by Taylor's Theorem,

$$\frac{f(x+h)}{\phi(x+h)}=\frac{f(x)+f'(x)h+f''(x)\frac{h^2}{\lfloor 2}+f'''(x)\frac{h^3}{\lfloor 3}+\cdots}{\phi(x)+\phi'(x)h+\phi''(x)\frac{h^2}{\lfloor 2}+\phi'''(x)\frac{h^3}{\lfloor 3}+\cdots}. \quad (1)$$

Substituting a for x, making $f(a)=0$ and $\phi(a)=0$, and dividing both terms of the fraction by h,

$$\frac{f(a+h)}{\phi(a+h)}=\frac{f'(a)+f''(a)\frac{h}{\lfloor 2}+f'''(a)\frac{h^2}{\lfloor 3}+\cdots}{\phi'(a)+\phi''(a)\frac{h}{\lfloor 2}+\phi'''(a)\frac{h^2}{\lfloor 3}+\cdots}. \quad (2)$$

Hence, as h approaches zero, by Art. 5,

$$\frac{f(a)}{\phi(a)}=\frac{f'(a)}{\phi'(a)}, \quad (3)$$

which is the true value of $\frac{f(x)}{\phi(x)}$ when $x=a$.

If $f'(a)=0$, and $\phi'(a)=0$, then $\frac{f'(a)}{\phi'(a)}=\frac{0}{0}$, and the result is still indeterminate. In this case, dropping the first term of the numerator

and also of the denominator of (2), dividing both terms of the fraction by $\frac{h}{\underline{|2}}$, as h approaches zero,

$$\frac{f(a)}{\phi(a)} = \frac{f''(a)}{\phi''(a)}. \tag{4}$$

If (4) is also indeterminate, the process is repeated until a ratio of two derivatives is obtained, both of which do not reduce to zero when $x = a$.

If $f^n(a) = 0$ and $\phi^n(a)$ be not 0, the true value is 0.

If $f^n(a)$ be not 0 and $\phi^n(a) = 0$, the true value is ∞.

PROBLEMS.

1. Find the true value of $\frac{x^5 - 1}{x - 1}$, when $x = 1$.

$$f(x) = x^5 - 1, \; \phi(x) = x - 1;$$

hence, $f'(x) = 5x^4$, and $\phi'(x) = 1$;

therefore $\frac{f(x)}{\phi(x)} = \frac{f'(x)}{\phi'(x)} = \frac{5x^4}{1} = 5$, when $x = 1$.

2. Find the true value of $\frac{x - \sin x}{x^3}$, when $x = 0$.

$$\frac{f(x)}{\phi(x)} = \frac{x - \sin x}{x^3} = \frac{0}{0}, \text{ when } x = 0;$$

$$\frac{f'(x)}{\phi'(x)} = \frac{1 - \cos x}{3x^2} = \frac{0}{0}, \text{ when } x = 0;$$

$$\frac{f''(x)}{\phi''(x)} = \frac{\sin x}{6x} = \frac{0}{0}, \text{ when } x = 0;$$

$$\frac{f'''(x)}{\phi'''(x)} = \frac{\cos x}{6} = \frac{1}{6}, \text{ when } x = 0;$$

therefore $\left[\frac{x - \sin x}{x^3}\right]_{x=0} = \frac{1}{6}.$

The subscript denotes the value which is to be substituted for x in the function within the brackets.

3. $\dfrac{\log x}{x-1}$, when $x=1$. *Ans.* 1.

4. $\dfrac{a^x-b^x}{x}$, when $x=0$. *Ans.* $\log\dfrac{a}{b}$.

5. $\dfrac{e^x-e^{-x}}{\sin x}$, when $x=0$. *Ans.* 2.

6. $\dfrac{1-x^n}{1-x}$, when $x=1$. *Ans.* n.

7. $\dfrac{\tan x+\sec x-1}{\tan x-\sec x+1}$, when $x=0$. *Ans.* 1.

8. $\dfrac{e^x-e^{-x}-2x}{x-\sin x}$, when $x=0$. *Ans.* 2.

9. $\dfrac{x^3-ax^2-a^2x+a^3}{x^2-a^2}$, when $x=a$. *Ans.* 0.

10. $\dfrac{ax-x^2}{a^4-2a^3x+2ax^3-x^4}$, when $x=a$. *Ans.* $-\infty$.

11. $\dfrac{\tan x-\sin x}{\sin^3 x}$, when $x=0$. *Ans.* $\frac{1}{2}$.

ART. 60. FUNCTIONS THAT TAKE THE FORM $\dfrac{\infty}{\infty}$.

When $x=a$, a function may take any one of the forms

$$\frac{\infty}{a},\quad \frac{a}{\infty},\quad \text{or}\quad \frac{\infty}{\infty}.$$

Evidently, $\dfrac{\infty}{a}=\infty$, $\dfrac{a}{\infty}=0$, and $\dfrac{\infty}{\infty}$ is indeterminate.

Let
$$\frac{f(x)}{\phi(x)}=\frac{\infty}{\infty},\ \text{when } x=a.$$

By taking the reciprocals of $f(x)$ and $\phi(x)$,

$$\frac{f(x)}{\phi(x)}=\frac{\dfrac{1}{\phi(x)}}{\dfrac{1}{f(x)}}=\frac{0}{0},\ \text{when } x=a.$$

Therefore, by Art. 59,

$$\frac{f(x)}{\phi(x)}=\frac{\dfrac{1}{\phi(x)}}{\dfrac{1}{f(x)}}=\frac{\dfrac{d}{dx}\left(\dfrac{1}{\phi(x)}\right)}{\dfrac{d}{dx}\left(\dfrac{1}{f(x)}\right)}=\frac{-\dfrac{\phi'(x)}{[\phi(x)]^2}}{-\dfrac{f'(x)}{[f(x)]^2}}=\frac{\phi'(x)[f(x)]^2}{f'(x)[\phi(x)]^2},$$

and when $x=a$,
$$\frac{f(a)}{\phi(a)}=\frac{\phi'(a)[f(a)]^2}{f'(a)[\phi(a)]^2}; \tag{1}$$

hence
$$\frac{f(a)}{\phi(a)}=\frac{f'(a)}{\phi'(a)}. \tag{2}$$

Therefore, the true value of the indeterminate form $\frac{\infty}{\infty}$ can be found by the same method as that of the form $\frac{0}{0}$.

However, in dividing (1) by $\frac{f(a)}{\phi(a)}$, it was assumed that $\frac{f(a)}{\phi(a)}$ is not equal to 0 or ∞. But (2) gives the true value in these cases also, as may be shown as follows:

Suppose the true value of $\frac{f(a)}{\phi(a)}$ to be 0, and let h be a finite quantity; then

$$\frac{f(a)}{\phi(a)}+h=\frac{f(a)+h\phi(a)}{\phi(a)}=h.$$

But $\frac{f(a)+h\phi(a)}{\phi(a)}$ has a value which is neither 0 nor ∞, hence (2) will apply to it, giving

$$\frac{f(a)+h\phi(a)}{\phi(a)}=\frac{f'(a)+h\phi'(a)}{\phi'(a)},$$

or
$$\frac{f(a)}{\phi(a)}+h=\frac{f'(a)}{\phi'(a)}+h,$$

therefore
$$\frac{f(a)}{\phi(a)}=\frac{f'(a)}{\phi'(a)}.$$

Similarly, if the true value of $\frac{f(a)}{\phi(a)}=\infty$ when $x=a$, then $\frac{\phi(a)}{f(a)}=0$, and the same demonstration applies.

ART. 61. FUNCTIONS THAT TAKE THE FORMS $0\times\infty$ AND $\infty-\infty$.

Let
$$f(x)\times\phi(x)=0\times\infty, \text{ when } x=a.$$

Then
$$f(a)\times\phi(a)=\frac{f(a)}{\dfrac{1}{\phi(a)}}=\frac{0}{0};$$

therefore, the true value may be found as in Art. 59.

Again, let $\quad f(x) - \phi(x) = \infty - \infty$, when $x = a$.

The expression in this case can be transformed into a fraction, which will assume either the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and the true value is found as before.

For example, to find the value of

$$\sec x - \tan x, \text{ when } x = \frac{\pi}{2}.$$

$$\sec x - \tan x = \frac{1 - \sin x}{\cos x} = \frac{0}{0}, \text{ when } x = \frac{\pi}{2}.$$

Therefore
$$\left[\frac{f'(x)}{\phi'(x)}\right]_{\frac{\pi}{2}} = \left[\frac{-\cos x}{-\sin x}\right]_{\frac{\pi}{2}} = 0.$$

ART. 62. FUNCTIONS THAT TAKE THE FORMS 0^0, ∞^0, AND $1^{\pm\infty}$.

Let $f(x)$ and $\phi(x)$ be two functions of x, which, when $x = a$, take such values that $[f(x)]^{\phi x}$ is one of the assumed forms.

Let
$$y = [f(x)]^{\phi(x)};$$
then
$$\log y = \phi(x) \log f(x). \tag{1}$$

1st. When $f(x) = \infty$ or 0, and $\phi(x) = 0$, (1) becomes

$$\phi(x) \log f(x) = 0\,(\pm\infty).$$

2d. When $f(x) = 1$, and $\phi(x) = \pm\infty$, (1) becomes

$$\phi(x) \log f(x) = (\pm\infty) \times 0.$$

Therefore, the true values of the logarithms of all the functions which take the forms, 0^0, ∞^0, and $1^{\pm\infty}$, may be obtained as in Art. 61.

For example, to find the value of x^x when $x = 0$.

Let
$$y = x^x;$$
then
$$\log y = x \log x = 0\,(-\infty), \text{ when } x = 0.$$
Hence,
$$\log y = \frac{\log x}{x^{-1}} = -\frac{\infty}{\infty}, \text{ when } x = 0.$$

$$\frac{\frac{d}{dx}(\log x)}{\frac{d}{dx}(x^{-1})} = \frac{\frac{1}{x}}{-x^{-2}} = [-x]_0 = 0;$$

therefore $\quad \log x^x = 0$, when $x = 0$; hence $x^x = 1$, when $x = 0$.

PROBLEMS.

Find the true values of the following functions:

1. $\dfrac{\log x}{x^a}$, when $x = \infty$. *Ans.* 0.

2. $\dfrac{\tan x}{3x}$, when $x = \frac{1}{2}\pi$. *Ans.* ∞.

3. $\dfrac{1 - \log x}{e^{\frac{1}{x}}}$, when $x = 0$. *Ans.* 0.

4. $\dfrac{\log \tan 2x}{\log \tan x}$, when $x = 0$. *Ans.* 1.

5. $x^n \log x$, when $x = 0$. *Ans.* 0.

6. $2^x \sin \dfrac{a}{2^x}$, when $x = \infty$. *Ans.* a.

7. $\sec x\left(x \sin x - \dfrac{\pi}{2}\right)$, when $x = \dfrac{\pi}{2}$. *Ans.* -1.

8. $\dfrac{2}{x^2 - 1} - \dfrac{1}{x - 1}$, when $x = 1$. *Ans.* $-\frac{1}{2}$.

9. $\dfrac{x}{\log x} - \dfrac{1}{\log x}$, when $x = 1$. *Ans.* 1.

10. $\operatorname{cosec}^2 x - \dfrac{1}{x^2}$, when $x = 0$. *Ans.* $\frac{1}{3}$.

11. $\dfrac{2}{\sin^2 x} - \dfrac{1}{1 - \cos x}$, when $x = 0$. *Ans.* $\frac{1}{2}$.

12. $\left(\dfrac{1}{x}\right)^{\tan x}$, when $x = 0$. *Ans.* 1.

13. $x^{\sin x}$, when $x = 0$. *Ans.* 1.

14. $(e^x + 1)^{\frac{1}{x}}$, when $x = \infty$. *Ans.* e.

15. $\left(1 + \dfrac{a}{x}\right)^x$, when $x = \infty$. *Ans.* e^a.

16. $\sin x^{\tan x}$, when $x = \dfrac{\pi}{2}$. *Ans.* 1.

ART. 63. COMPOUND INDETERMINATE FORMS.

When a given function can be resolved into factors, one or more of which become indeterminate for a particular value of x, the true value may be obtained by getting the true value of each factor separately.

When the true value of any indeterminate form is found, that of any constant power of it can be determined.

PROBLEMS.

1. $\dfrac{x^a - x^{a+c}}{1 - x^{2b}}$, when $x = 1$.

This may be put in the form $\dfrac{x^a}{1 + x^b} \cdot \dfrac{1 - x^c}{1 - x^b}$,

in which the second factor only is indeterminate.

Let $$\frac{f(x)}{\phi(x)} = \frac{1 - x^c}{1 - x^b};$$

then $$\frac{f'(x)}{\phi' x} = \frac{-cx^{c-1}}{-bx^{b-1}} = \frac{c}{b} x^{c-b} = \frac{c}{b}, \text{ when } x = 1.$$

Therefore $$\left[\frac{x^a - x^{a+c}}{1 - x^{2b}}\right]_1 = \frac{c}{2b}.$$

2. $\dfrac{(e^x - 1)\tan^2 x}{x^3}$, when $x = 0$.

$$\frac{(e^x - 1)\tan^2 x}{x^3} = \left(\frac{\tan x}{x}\right)^2 \frac{(e^x - 1)}{x}.$$

$$\left[\frac{\tan x}{x}\right]_0 = 1, \text{ and } \left[\frac{e^x - 1}{x}\right]_0 = 1;$$

therefore $$\frac{(e^x - 1)\tan^2 x}{x^3} = 1, \text{ when } x = 1.$$

3. $\dfrac{\log(1 + x + x^2) + \log(1 - x + x^2)}{\sec x - \cos x}$, when $x = 0$. *Ans.* 1.

4. $\dfrac{(x^2 - a^2)\sin\dfrac{\pi x}{2a}}{x^2 \cos\dfrac{\pi x}{2a}}$, when $x = a$. *Ans.* $\dfrac{1}{a^2}$.

5. $x^m (\sin x)^{\tan x}\left(\dfrac{\pi - 2x}{2\sin 2x}\right)^3$, when $x = \dfrac{\pi}{2}$. *Ans.* $\dfrac{\pi^m}{2^{m+3}}$.

CHAPTER X.

MAXIMA AND MINIMA OF FUNCTIONS.

ART. 64. DEFINITIONS AND GEOMETRIC ILLUSTRATION.

A *maximum* value of a function is a certain value at which the function changes from an increasing to a decreasing function. Or, in other words, $f(x)$ is a maximum for that value of x which makes $f(x)$ greater than $f(x+h)$ and $f(x-h)$ for very small values of h.

A *minimum* value of a function is a certain value at which the function changes from a decreasing to an increasing function. Or, $f(x)$ is a minimum for that value of x which makes $f(x)$ less than $f(x+h)$ and $f(x-h)$ for very small values of h.

In Fig. 8, let the curve AB be the locus of $y=f(x)$.

Then PN represents a maximum ordinate, and $P'T$ a minimum ordinate. As x increases toward ON, y approaches a maximum value,

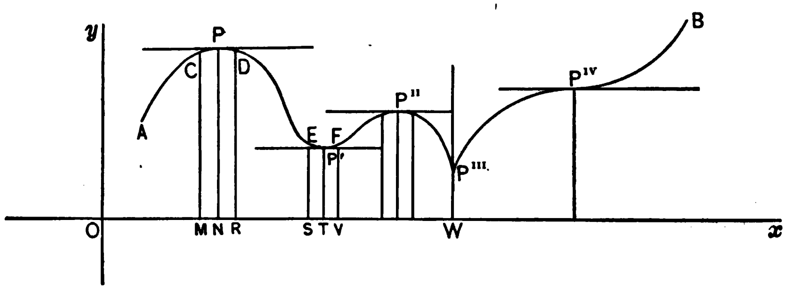

FIG. 8.

PN, and the tangent to the curve makes an acute angle with the X-axis. At the point P, the tangent line is parallel to the X-axis. Immediately after passing P, the tangent makes an obtuse angle with the X-axis. But by Art. 27, the slope of the tangent line is equal

to $f'(x)$; hence $f'(x)$ is positive before, and negative after, a maximum ordinate. Likewise it may be shown that $f'(x)$ is negative before, and positive after, a minimum ordinate. Thus, $f'(x)=0$, at both maximum and minimum ordinates. Therefore, a condition for both maxima and minima is $\frac{dy}{dx}=0$.

ART. 65. METHOD OF DETERMINING MAXIMA AND MINIMA.

For a maximum value of the function, $f'(x)=0$ and $f'(x)$ changes sign from $+$ to $-$ when x passes through a value corresponding to a maximum value of the function. For a minimum value of the function, $f'(x)=0$ and $f'(x)$ changes sign from $-$ to $+$ when x passes through a value corresponding to a minimum value of the function. Hence, the roots of the equation $f'(x)=0$ are first obtained.

If a is a root of this equation, a value slightly less, and then one slightly greater than a, are substituted for x in $f'(x)$. Let h represent a very small quantity.

If $f'(a-h)$ is $+$, and $f'(a+h)$ is $-$, then $f(a)$ is a maximum.

If $f'(a-h)$ is $-$, and $f'(a+h)$ is $+$, then $f(a)$ is a minimum.

For example, let $y=b+(x-a)^4$.

Then $$\frac{dy}{dx}=4(x-a)^3=0;\text{ hence } x=a.$$

Substituting $a-h$ for x, gives

$$\frac{dy}{dx}=4(a-h-a)^3=-4h^3.$$

Substituting $a+h$ for x, gives

$$\frac{dy}{dx}=4(a+h-a)^3=+4h^3.$$

Here, $f'(x)$ changes sign from $-$ to $+$ at $x=a$; hence, a is the value of x which gives a minimum function. Therefore, $y=b$, a minimum.

By reference to Fig. 8, it will be seen that $P'''W$ is a minimum ordinate, and the tangent to the curve at this point is perpendicular to

the X-axis. In this case $f'(x)$ changes sign by passing through ∞. Any variable can change its sign only by passing through 0 or ∞, but it does not necessarily follow that there is a change of sign whenever $f'(x)=0$, or $f'(x)=\infty$. At point P^{IV}, the tangent is parallel to the X-axis, hence $f'(x)=0$; but $f'(x)$ is $+$ immediately before and after reaching this value. Therefore, the values of x which make $f'(x)=0$ do not always give maxima and minima, so they are simply called *critical* values, or values for which the function is to be examined.

It is evident also that a function may have several maxima and minima, and a minimum value may be greater than a maximum value of the same function.

ART. 66. CONDITIONS FOR MAXIMA AND MINIMA BY TAYLOR'S THEOREM.

Let $f(x)$ have a maximum or minimum value when $x=a$.

Then if h be a very small increment of x, by Art. 64,

$$f(a)>f(a+h), \text{ and } f(a)>f(a-h), \text{ for a maximum,}$$

also $$f(a)<f(a+h), \text{ and } f(a)<f(a-h), \text{ for a minimum.}$$

Therefore $$f(a+h)-f(a) \text{ and } f(a-h)-f(a)$$

are each negative for a maximum, or are each positive for a minimum.

Now by Taylor's Theorem,

$$f(a+h)-f(a)=\quad f'(a)h+f''(a)\frac{h^2}{\underline{|2}}+f'''(a)\frac{h^3}{\underline{|3}}+\cdots. \qquad (1)$$

$$f(a-h)-f(a)=-f'(a)h+f''(a)\frac{h^2}{\underline{|2}}-f'''(a)\frac{h^3}{\underline{|3}}+\cdots. \qquad (2)$$

For a maximum: The first members of (1) and (2) must be negative, therefore the second members must be negative. Now if h be taken sufficiently small, the first term in each second member can be made numerically greater than the sum of all the terms following it; hence, the sign of each second member will be the same as that of its first term. But the first terms have different apparent signs, so the

second members cannot both be negative unless the first term disappears, hence

$$f'(a) = 0.$$

Now the first of the remaining terms of the second members contain h^2, and these terms determine the signs of the members. In order that these terms may be negative, $f''(a)$ must be negative, or

$$f''(a) < 0.$$

Therefore, if $f(a)$ is a maximum,

$$f'(a) = 0 \text{ and } f''(a) < 0.$$

Similarly, it may be shown that if $f(a)$ is a minimum,

$$f'(a) = 0 \text{ and } f''(a) > 0.$$

However, if $f''(a) = 0$, then the sign of the second members of (1) and (2) will depend on the terms containing $f'''(a)$, and since the terms containing $f'''(a)$ have opposite signs, there can be neither a maximum nor a minimum unless $f'''(a)$ also vanishes; and if $f'''(a) = 0$, then $f(a)$ is a maximum when $f^{\text{IV}}(a)$ is negative, and a minimum when $f^{\text{IV}}(a)$ is positive; and so on.

RULE: *Find $f'(x)$, and solve the equation, $f'(x) = 0$. Substitute the roots of this equation for x in $f''(x)$. Each value of x which makes $f''(x)$ negative will make $f(x)$ a maximum; and each value which makes $f''(x)$ positive will make $f(x)$ a minimum.*

However, if any value of x also makes $f''(x) = 0$, substitute this value in the successive derivatives until one does not reduce to 0. If this be of an odd order, the value of x will give neither a maximum nor a minimum; but if it be of an even order and negative, $f(x)$ will be a maximum, if of an even order and positive, $f(x)$ will be a minimum.

The solution of problems in maxima and minima is often simplified by the aid of the following principles:

1. If any value of x makes $af(x)$ a maximum or minimum, a being a positive constant, that value will make $f(x)$ a maximum or minimum. Hence, a constant factor may be omitted.

2. If any value of x makes $[f(x)]^n$ a maximum or minimum, n being a positive constant, that value will make $f(x)$ a maximum or minimum.

Hence, any constant exponent of the function may be omitted; or if the function is a radical, the radical sign may be omitted.

3. If any value of x makes $\log f(x)$ a maximum or minimum, that value will make $f(x)$ a maximum or minimum. Hence, to find a maximum or minimum value of the logarithm of a function, the function only need be taken.

4. If any value of x makes $f(x)$ a maximum or minimum, that value will make $\frac{1}{f(x)}$ a minimum or maximum. Hence, when a function is a maximum or a minimum, its reciprocal is a minimum or a maximum.

5. If any value of x makes $a + f(x)$ a maximum or minimum, that value will make $f(x)$ a maximum or minimum. Hence, a constant term may be omitted.

Each of the preceding propositions may be readily proved. For example, in (1), the first derivative of $af(x)$ when placed equal to zero, will give an equation whose roots are the same as the roots of the equation formed by placing the first derivative of $f(x)$ equal to zero; hence, the critical values will be the same in both cases.

PROBLEMS.

1. Find the maximum and minimum values of

$$x^3 - 3x^2 - 9x + 5.$$

Let $$y = x^3 - 3x^2 - 9x + 5;$$

then $$\frac{dy}{dx} = 3x^2 - 6x - 9.$$

Placing the first derivative equal to zero, and finding the roots,

$$3x^2 - 6x - 9 = 0;$$

therefore $$x = 3 \text{ or } -1.$$

The second derivative is $\frac{d^2y}{dx^2} = 6x - 6.$

When $x=3$, $\frac{d^2y}{dx^2}=12$, and as this value of x makes the second derivative positive, it corresponds to a minimum value of the function.

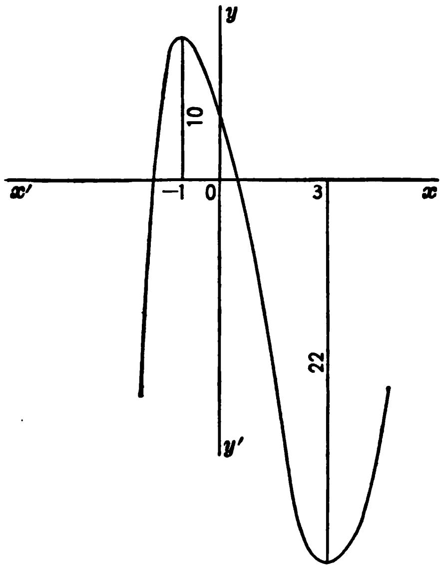

FIG. 9.

When $x=-1$, $\frac{d^2y}{dx^2}=-12$, and this result being negative indicates a maximum.

Substituting these values of x in the function, gives, when $x=3$, $y=-22$, a minimum, and when $x=-1$, $y=10$, a maximum.

These results may be illustrated graphically by constructing the locus of the equation.

In Fig. 9 it will be seen that there is a maximum ordinate corresponding to the abscissa -1, and a minimum ordinate corresponding to the abscissa 3.

REMARK. It will be very instructive to construct the loci of the equations in the first few examples.

Examine the following functions for Maxima and Minima:

2. $y=x^5-5x^4+5x^3+1$. *Ans.* $x=1$, gives a Maximum, 2;
$x=3$, gives a Minimum, -26.

3. $y=2x^3-21x^2+36x-20$. *Ans.* $x=1$, gives Max., -3;
$x=6$, gives Min., -128.

4. $y=3x^3-9x^2-27x+30$. *Ans.* $x=-1$, gives Max., 45;
$x=3$, gives Min., -51.

5. $y=\dfrac{(a-x)^3}{a-2x}$. *Ans.* $x=\frac{1}{4}a$, gives Min., $\frac{27}{32}a^2$.

6. $y = x^3 - 3x^2 + 6x + 10.$

Ans. This function has no real Max. or Min.

7. $y = \dfrac{x}{1 + x\tan x}.$ *Ans.* $x = \cos x$, gives a Max.

8. $y = x^{\frac{1}{x}}.$ *Ans.* $x = e$, gives Max.

9. $y = \dfrac{\sin x}{1 + \tan x}.$ *Ans.* $x = 45°$, gives Max.

10. $y = \sin x(1 + \cos x).$ *Ans.* $x = \dfrac{\pi}{3}$, gives Max.

11. $y = (x - 1)^4 (x + 2)^3.$ *Ans.* $x = -\frac{5}{7}$, gives Max.;

$x = 1$, gives Min.;

$x = -2$, gives neither.

GEOMETRIC PROBLEMS.

12. Determine the maximum rectangle inscribed in a given circle.

Assume an inscribed rectangle as in Fig. 10. Let the diameter $CB = d$, and the side $CD = x$; then

$$AC = \sqrt{d^2 - x^2}.$$

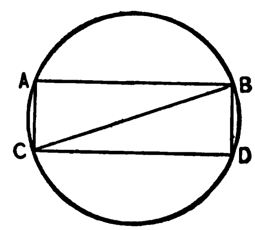

FIG. 10.

Denoting the area by A, then

$$A = x\sqrt{d^2 - x^2},$$

which is to be a maximum.

By Art. 66, 2, the function $x^2(d^2 - x^2)$ may be used.

Put $y = x^2(d^2 - x^2).$

Then $\dfrac{dy}{dx} = 2d^2x - 4x^3 = 0$; hence $x = 0$, or $x = d\sqrt{\frac{1}{2}}.$

Now $\dfrac{d^2y}{dx^2} = 2d^2 - 12x^2 = 2d^2$, when $x = 0$;

$= -4d^2$, when $x = d\sqrt{\frac{1}{2}}.$

Therefore, $x = d\sqrt{\frac{1}{2}}$, which is the side of an inscribed square, will give the maximum rectangle.

13. Find the greatest cylinder which can be inscribed in a given right cone with a circular base.

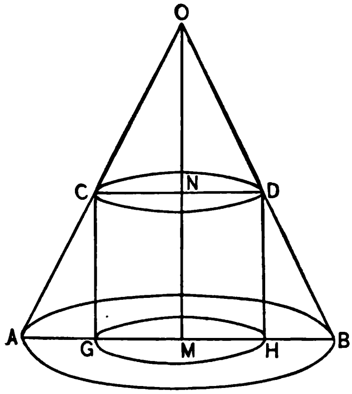

FIG. 11.

In Fig. 11, let CH be a cylinder inscribed in the cone OAB.

Given $AM = a$, and $OM = h$.

Let $NC = x$, and $NM = y$.

Denoting the volume by V,

then $V = \pi x^2 y$.

From the similar triangles AOM and CON,

$$\frac{h}{a} = \frac{h-y}{x}, \text{ hence } x = \frac{a}{h}(h-y).$$

Therefore, $V = \pi \frac{a^2}{h^2}(h-y)^2 y$, which is found to be a maximum when $y = \frac{1}{3}h$. Therefore, the altitude of the maximum inscribed cylinder is one-third of the altitude of the cone.

14. Find the maximum cone which can be inscribed in a sphere whose radius is r.

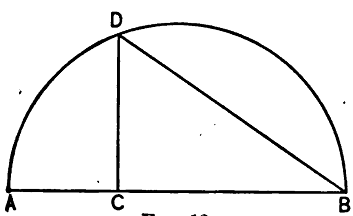

FIG. 12.

In Fig. 12, let ADB and CDB be the semicircle and triangle which generate the sphere and inscribed cone by revolution about AB.

Let $CD = x$, $CB = y$, and $V =$ the volume of the cone;

then $V = \frac{1}{3}\pi x^2 y$.

$$x^2 = CB \times CA = y(2r - y),$$

hence, $V = \frac{1}{3}\pi y^2(2r - y)$, which is the function whose maximum is required. *Ans.* The altitude of the Max. cone $= \frac{4}{3}r$.

15. Determine the right cylinder of greatest convex surface that can be inscribed in a given sphere.

If $r =$ the radius of the sphere, and $x =$ the radius of the base of the

cylinder, then the convex surface of the cylinder is $4\pi x\sqrt{r^2 - x^2}$. This will be a maximum when the radius of the base is $\frac{r}{\sqrt{2}}$.

16. From a given surface S, a cylindrical vessel with circular base and open top is to be made, so as to contain the greatest amount. To find its dimensions.

Let $x =$ radius of base, $y =$ altitude, and $V =$ volume of a cylinder.

Then $$V = \pi x^2 y, \tag{1}$$

and $$S = \pi x^2 + 2\pi xy. \tag{2}$$

Differentiating (1) and (2) with respect to x:

From (1), $$\frac{dV}{dx} = 2\pi xy + \pi x^2 \frac{dy}{dx} = 0,$$

hence $$\frac{dy}{dx} = -\frac{2y}{x}.$$

From (2), $$0 = 2\pi x + 2\pi x\frac{dy}{dx} + 2\pi y,$$

hence $$\frac{dy}{dx} = -\frac{x+y}{x}.$$

Hence $$-\frac{2y}{x} = -\frac{x+y}{x}.$$

Therefore $y = x$, or the altitude $=$ radius of base.

In this example, (2) might have been solved for y and this value substituted in (1), and the solution would have been the usual one. But the given solution is in this and similar examples much shorter.

17. What is the length of the axis of the maximum parabola which can be cut from a given right circular cone, knowing that the area of a parabola is equal to two-thirds of the product of its base and altitude?

Given $BC = a$, and $AB = b$, in Fig. 13.

Let $CM = x$, then $BM = a - x$,

and $RS = 2\sqrt{(a-x)x}$.

By similar triangles,

$$a:b::x:MN; \quad \therefore \quad MN = \frac{b}{a}x.$$

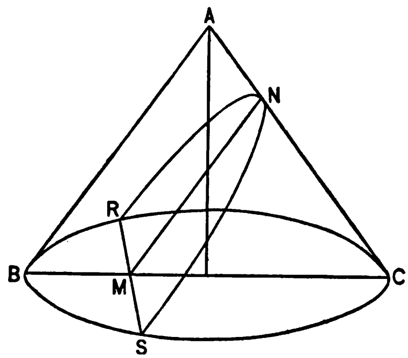

FIG. 13.

Hence, the area of the parabola is

$$A = \frac{4}{3}\frac{b}{a}x\sqrt{(a-x)x},$$

which is a maximum when $x = \frac{3}{4}a$.

18. What is the altitude of the maximum rectangle which can be inscribed in a given segment of a parabola?

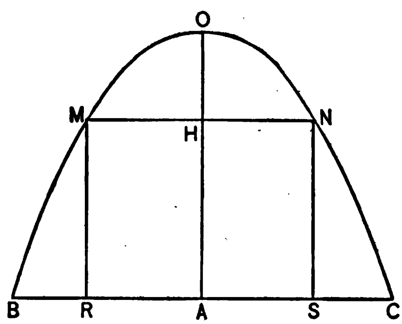

FIG. 14.

In Fig. 14, let BOC be the parabolic segment and $AO = h$.

Let $$OH = x,$$

then $$MH = \sqrt{2px}.$$

Therefore, area of $MRSN = 2\sqrt{2px}(h - x)$,

which is a maximum when $x = \frac{h}{3}$.

19. What is the maximum cylinder that can be inscribed in an oblate spheroid whose semi-axes are a and b?

Ans. Radius of base $= \frac{1}{3}a\sqrt{6}$; altitude $= \frac{2}{3}b\sqrt{3}$.

20. Find the maximum right cone that can be inscribed in a given right cone, the vertex of the required cone being at the centre of the base of the given cone. *Ans.* The ratio of the altitudes is $\frac{1}{3}$.

21. What is the maximum isosceles triangle which can be inscribed in a circle? *Ans.* An equilateral triangle.

22. What is the altitude of the cone of maximum convex surface that can be inscribed in a sphere whose radius is 3?

Ans. Altitude $= 4$.

23. When is the difference between the sine and the cosine of any angle a maximum? *Ans.* When the angle $= 135°$.

24. If the strength of a beam with rectangular cross-section varies directly as the breadth, and as the square of the depth, what are the dimensions of the strongest beam that can be cut from a round log whose diameter is D? *Ans.* Depth $= D\sqrt{\frac{2}{3}}$.

25. A rectangular box with a square base and open at the top, is to be constructed to contain 108 cubic inches. What must be its dimensions so as to contain the least material?

Ans. Altitude $= 3$ inches; side of base $= 6$ inches.

26. What is the altitude of the minimum cone that may be circumscribed about a sphere whose diameter is 10? *Ans.* Altitude $= 20$

27. A person, being in a boat 3 miles from the nearest point of the beach, wishes to reach in the shortest time a place 5 miles from that point along the shore; supposing he can walk 5 miles an hour, but can row only at the rate of 4 miles per hour, required the place he must land. *Ans.* One mile from the place to be reached.

28. Find the minimum value of y when $y = \frac{x^3 + 2b^3}{3(x^3 + b^2 x)}$.

Ans. $y = 0.32218$.

29. Determine the greatest rectangle which can be inscribed in a given triangle whose base $= 2b$ and altitude $= a$. *Ans.* $A = \frac{1}{2}ab$.

30. A Norman window consists of a rectangle surmounted by a semicircle. Given the perimeter, required the height and breadth of the window when the quantity of light admitted is a maximum.

Ans. Radius of semicircle = height of rectangle.

31. A privateer must pass between two lights A and B on opposite headlands, the distance between which is c. The intensity of light A at a unit's distance is a, and the intensity of light B at a unit's distance is b. At what point must a privateer pass the line joining the lights so as to be as little in the light as possible, assuming the principle of optics, that the intensity of a light at any point is equal to its intensity at a unit's distance divided by the square of the distance of the point from the light?

Ans. $x = \frac{ca^{\frac{1}{3}}}{a^{\frac{1}{3}} + b^{\frac{1}{3}}}$.

32. The flame of a candle is directly over the centre of a circle whose radius is 5; what ought to be its height above the plane of the circle so as to illuminate the circumference as much as possible, supposing the intensity of the light to vary directly as the sine of the angle under which it strikes the illuminated surface, and inversely as the square of its distance from the same surface?

Ans. Height above circle $= 5\sqrt{\frac{1}{2}}$.

33. If the total waste per mile in an electric conductor is $C^2 r + \frac{t^2}{r}$ (r ohms resistance per mile), due to heat, interest, and depreciation, what is the relation between C, r and t when the waste is a minimum?

Ans. $Cr = t$.

34. In the formula, $A^2 B = \frac{\rho^4 (x^2 + x^4) d^2 c}{E^2 (x-1)^2}$, it is required to find the value of x that makes the variable factor $\frac{x^2 + x^4}{(x-1)^2}$ a minimum.

Ans. $x = 2.2$.

35. From the differential equation, $20{,}000{,}000 \dfrac{d^2y}{dx^2} = -100x$, find the equation of the curve and the maximum ordinate; the first constant of integration being found by making $\dfrac{dy}{dx} = 0$ when x equals l, and the second constant of integration being found by making y equal to zero when x equals zero.

Ans. $y = \dfrac{1}{20{,}000{,}000}\left(\dfrac{100\,l^2x}{2} - \dfrac{100\,x^3}{6}\right)$, and Max. ordinate $= \dfrac{l^3}{600{,}000}$.

36. A statue whose height is 10 feet stands on a pedestal 8 feet in height, which rests on a level surface. At what point on the horizontal plane through the base of the pedestal does the statue subtend the greatest angle? *Ans.* 12 feet from centre of base.

37. If v denotes the velocity of a current, and x the velocity of a steamer through the water against the current, what will be the speed most economical in fuel if the quantity of fuel burnt is proportional to x^3? *Ans.* $x = \frac{3}{2}v$.

ART. 67. MAXIMA AND MINIMA OF FUNCTIONS OF TWO INDEPENDENT VARIABLES.

Let $f(x, y)$ represent any function of two independent variables.

When $$f(x, y) > f(x+h, y+k),$$

for all very small values of h and k, positive or negative, the function has a maximum value.

When $$f(x, y) < f(x+h, y+k),$$

for such values of h and k, the function has a minimum value.

Placing $$u = f(x, y),$$

from Art. 57,

$$f(x+h, y+k) - f(x, y) = h\frac{\partial u}{\partial x} + k\frac{\partial u}{\partial y} + \frac{1}{\underline{|2}}\left[h^2\frac{\partial^2 u}{\partial x^2} + 2hk\frac{\partial^2 u}{\partial x\partial y} + k^2\frac{\partial^2 u}{\partial y^2}\right] + \cdots \quad (1)$$

Now, by argument similar to that of Art. 66, it may be proved that the sign of $f(x+h, y+k) - f(x, y)$ will depend on $h\dfrac{\partial u}{\partial x} + k\dfrac{\partial u}{\partial y}$, and

therefore will change sign with h and k; hence, a maximum or minimum value is impossible unless

$$h\frac{\partial u}{\partial x}+k\frac{\partial u}{\partial y}=0.$$

And since h and k are independent,

$$\frac{\partial u}{\partial x}=0, \text{ and } \frac{\partial u}{\partial y}=0. \tag{2}$$

Substituting $A=\frac{\partial^2 u}{\partial x^2}$, $B=\frac{\partial^2 u}{\partial x\,\partial y}$, and $C=\frac{\partial^2 u}{\partial y^2}$, in equation (1), gives

$$f(x+h, y+k)-f(x, y)=\frac{1}{\underline{|2}}(Ah^2+2\,Bhk+Ck^2)+\cdots$$

$$=\frac{1}{\underline{|2}}\frac{k^2}{A}\left[\left(A\frac{h}{k}+B\right)^2+(AC-B^2)\right]+\cdots \tag{3}$$

In (3), the sign of $f(x+h, y+k)-f(x, y)$ will depend on

$$\frac{k^2}{A}\left[\left(A\frac{h}{k}+B\right)^2+(AC-B^2)\right],$$

and in order that it may retain the same sign for all very small values of h and k, it is necessary that $AC-B^2$ should be positive; for if $AC-B^2$ be negative, it will be possible, by ascribing some suitable value to $\frac{h}{k}$ to make the whole expression change its sign. Hence as a condition for a maximum or minimum,

$$B^2<AC. \tag{4}$$

It is obvious from (4) that A and C will have the same sign, and the sign of (3) thus depends on A.

Hence, for a maximum, $A<0$, and $C<0$;

and for a minimum, $A>0$, and $C>0$.

Therefore, the conditions established are:

For either a maximum or minimum,

$$\frac{\partial u}{\partial x}=0,\ \frac{\partial u}{\partial y}=0, \text{ and } \left(\frac{\partial^2 u}{\partial y\,\partial x}\right)^2<\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}.$$

Also, for a maximum, $\dfrac{\partial^2 u}{\partial x^2} < 0$ and $\dfrac{\partial^2 u}{\partial y^2} < 0$,

and for a minimum, $\dfrac{\partial^2 u}{\partial x^2} > 0$ and $\dfrac{\partial^2 u}{\partial y^2} > 0$.

ART. 68. CONDITIONS FOR MAXIMA AND MINIMA OF FUNCTIONS OF THREE INDEPENDENT VARIABLES.

By an investigation similar to that of Art. 67, the following conditions for a maximum or minimum value of $u = f(x, y, z)$ are established:

For either a maximum or minimum,

$$\frac{\partial u}{\partial x} = 0,\ \frac{\partial u}{\partial y} = 0,\ \frac{\partial u}{\partial z} = 0,\ \left(\frac{\partial^2 u}{\partial x\,\partial y}\right)^2 < \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2},$$

$$\left(\frac{\partial^2 u}{\partial y\,\partial z}\right)^2 < \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial z^2},\ \text{and}\ \left(\frac{\partial^2 u}{\partial z\,\partial x}\right)^2 < \frac{\partial^2 u}{\partial z^2}\frac{\partial^2 u}{\partial x^2}.$$

Also, for a maximum, $\dfrac{\partial^2 u}{\partial x^2} < 0$, $\dfrac{\partial^2 u}{\partial y^2} < 0$, and $\dfrac{\partial^2 u}{\partial z^2} < 0$,

and for a minimum, $\dfrac{\partial^2 u}{\partial x^2} > 0$, $\dfrac{\partial^2 u}{\partial y^2} > 0$, and $\dfrac{\partial^2 u}{\partial z^2} > 0$.

PROBLEMS.

1. Find a point so situated that the sum of the squares of its distances from the three vertices of a given triangle shall be a minimum.*

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be the coördinates of the vertices, and (x, y) the given point.

Then,

$$[(x-x_1)^2 + (y-y_1)^2] + [(x-x_2)^2 + (y-y_2)^2] + [(x-x_3)^2 + (y-y_3)^2]$$

is the function to be a minimum, which may be represented by u.

$$\frac{\partial u}{\partial x} = 2(x-x_1) + 2(x-x_2) + 2(x-x_3),$$

$$\frac{\partial u}{\partial y} = 2(y-y_1) + 2(y-y_2) + 2(y-y_3),$$

* See Byerly's *Diff. Calc.*, p. 236.

$$\frac{\partial^2 u}{\partial x^2} = 2 + 2 + 2 = 6 = A,$$

$$\frac{\partial^2 u}{\partial x \, \partial y} = 0 = B,$$

$$\frac{\partial^2 u}{\partial y^2} = 2 + 2 + 2 = 6 = C.$$

Making $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ each equal to zero:

$$2(x - x_1) + 2(x - x_2) + 2(x - x_3) = 0,$$

therefore $$x = \frac{x_1 + x_2 + x_3}{3}.$$

$$2(y - y_1) + 2(y - y_2) + 2(y - y_3) = 0,$$

therefore $$y = \frac{y_1 + y_2 + y_3}{3}.$$

$$AC - B^2 = 36 - 0 > 0,$$

$$A = 6 > 0.$$

Hence, u is a minimum when

$$x = \frac{x_1 + x_2 + x_3}{3}, \text{ and } y = \frac{y_1 + y_2 + y_3}{3},$$

and the required point is the centre of gravity of the triangle.

2. Find the maximum value of $x^3 y^2 (6 - x - y)$.

Ans. Max. when $x = 3$, $y = 2$.

3. Find the maximum value of $3axy - x^3 - y^3$.

Ans. Max. when $x = a$, $y = a$.

4. What is the triangle of maximum perimeter that may be inscribed in a given circle?

Ans. An equilateral triangle.

5. Find the values of x, y and z, that make $x^2 + y^2 + z^2 + x - 2z - xy$ a minimum.

Ans. $x = -\frac{2}{3}$, $y = -\frac{1}{3}$, $z = 1$.

6. What rectangular parallelopiped inscribed in a given sphere has the maximum volume? *Ans.* A cube.

7. An open vessel is to be constructed in the form of a rectangular parallelopiped, capable of containing 108 cubic inches of water. What must be its dimensions to require the least material in construction?

Ans. Length and width, 6 inches; height, 3 inches.

8. Prove that the point, the sum of the squares of whose distances from n given points situated in the same plane shall be a minimum, is the centre of mean position of the given points.

CHAPTER XI.

TANGENTS, NORMALS AND ASYMPTOTES TO ANY PLANE CURVE.

ART. 69. EQUATIONS OF THE TANGENT AND NORMAL.

Let $y=f(x)$ be the equation of any plane curve AB, and (x', y') the coördinates of the point of tangency T, in Fig. 15. The equation of a

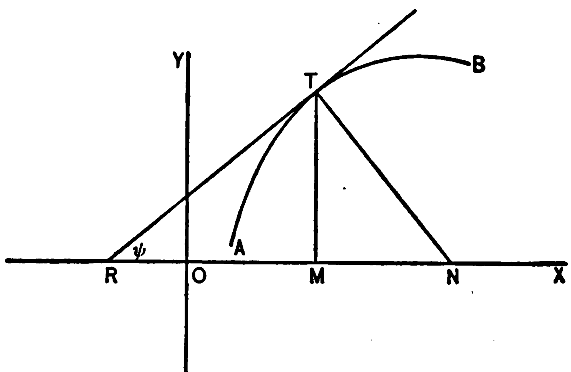

FIG. 15.

straight line through T is $y-y'=a(x-x')$, in which a is the tangent of the angle which the line makes with the X-axis.

By Art. 27, $$a=\tan\psi=\frac{dy'}{dx'}.$$

Therefore $$y-y'=\frac{dy'}{dx'}(x-x') \tag{1}$$

is the equation of the tangent to the curve $y=f(x)$ at the point (x', y').

As the normal to the curve at the point T is perpendicular to the tangent at that point, the slope of the normal is $-\frac{dx'}{dy'}$, and hence the equation of the normal is

$$y-y'=-\frac{dx'}{dy'}(x-x'). \tag{2}$$

ART. 70. LENGTHS OF TANGENT, NORMAL, SUBTANGENT AND SUBNORMAL.

In Fig. 15, let TM be the ordinate, TR the tangent and TN the normal, at the point of contact; then MR is the subtangent and MN the subnormal.

$$RM = \frac{TM}{\tan MRT} = y'\frac{dx'}{dy'};$$

hence

$$\text{subtangent} = y'\frac{dx'}{dy'}. \tag{3}$$

$$MN = MT \tan MTN = y'\frac{dy'}{dx'};$$

hence

$$\text{subnormal} = y'\frac{dy'}{dx'}. \tag{4}$$

$$RT = \sqrt{(MR)^2 + (MT)^2} = \sqrt{\left(y'\frac{dx'}{dy'}\right)^2 + y'^2};$$

hence

$$\text{tangent} = y'\sqrt{1 + \left(\frac{dx'}{dy'}\right)^2}. \tag{5}$$

$$TN = \sqrt{(MT)^2 + (MN)^2} = \sqrt{y'^2 + \left(y'\frac{dy'}{dx'}\right)^2};$$

hence

$$\text{normal} = y'\sqrt{1 + \left(\frac{dy'}{dx'}\right)^2}. \tag{6}$$

ART. 71. TANGENT OF THE ANGLE BETWEEN THE RADIUS VECTOR AT ANY POINT OF A PLANE CURVE AND THE TANGENT TO THE CURVE AT THAT POINT, IN POLAR COÖRDINATES.

Let O be the pole, OX the initial line, and P any point of the curve AB, in Fig. 16.

Let (r, θ) be the coördinates of P, and $(r + \Delta r, \theta + \Delta\theta)$ be the coördinates of another point R of the curve. If PS is perpendicular to OR, then

$$PS = r \sin \Delta\theta,$$

and

$$SR = (r + \Delta r) - r \cos \Delta\theta.$$

Therefore

$$\tan SRP = \frac{r \sin \Delta\theta}{r + \Delta r - r \cos \Delta\theta}.$$

Now, if the point R approaches P, then the secant RP approaches the tangent PT, and the angle SRP approaches the angle OPT. If the angle OPT is represented by ϕ, then

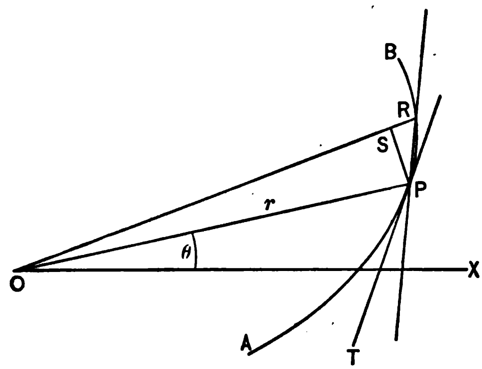

FIG. 16.

$$\tan\phi = \text{limit}\,\frac{r\sin\Delta\theta}{r+\Delta r - r\cos\Delta\theta}$$

$$= \text{limit}\,\frac{\dfrac{r\sin\Delta\theta}{\Delta\theta}}{\dfrac{2r\sin^2\frac{\Delta\theta}{2}}{\Delta\theta}+\dfrac{\Delta r}{\Delta\theta}}.$$

Now, $$\text{limit}\,\frac{\sin\Delta\theta}{\Delta\theta}=1;\quad \text{limit}\,\frac{\Delta r}{\Delta\theta}=\frac{dr}{d\theta};$$

$$\text{limit}\,\frac{2\sin^2\frac{\Delta\theta}{2}}{\Delta\theta} = \text{limit}\,\frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\sin\frac{\Delta\theta}{2}=0.$$

Therefore $$\tan\phi = r\frac{d\theta}{dr}. \tag{1}$$

ART. 72. DERIVATIVE OF AN ARC.

In Fig. 17, let P and P' be two points on the curve AB separated by a short distance Δs. The coördinates of P and P' are (x, y) and $(x+\Delta x, y+\Delta y)$ respectively.

In the right triangle PMP',

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2};$$

hence
$$\frac{\Delta s}{\Delta x} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}.$$

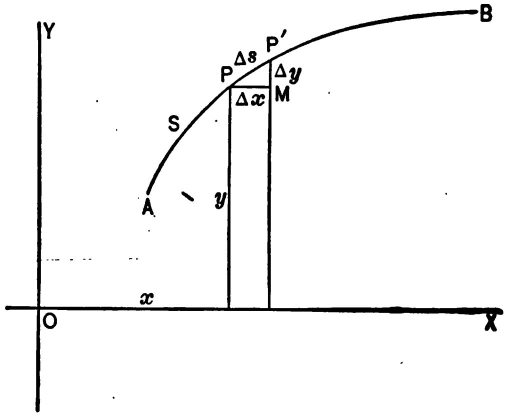

FIG. 17.

Now, when P' approaches P,

$$\text{limit}\,\frac{\Delta s}{\Delta x} = \text{limit}\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2},$$

or
$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

ART. 73. DERIVATIVE OF AN ARC IN POLAR COÖRDINATES.

From Fig. 16, regarding the limiting triangle of PSR,

$$\text{limit}\sec PRS = \text{limit}\,\frac{PR}{RS} = \text{limit}\,\frac{\Delta s}{\Delta r},$$

hence
$$\sec\phi = \frac{ds}{dr}. \qquad (1)$$

$$\frac{ds}{dr} = \sqrt{1 + \tan^2\phi} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}, \text{ by Art. 71, (1);}$$

therefore
$$\frac{ds}{d\theta} = \frac{ds}{dr}\,\frac{dr}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}. \qquad (2)$$

ART. 74. LENGTHS OF TANGENT, NORMAL, SUBTANGENT, SUBNORMAL AND PERPENDICULAR ON TANGENT, IN POLAR COÖRDINATES.

In Fig. 18, let PT be the tangent to the curve at the point P, and through P the normal PN is drawn.

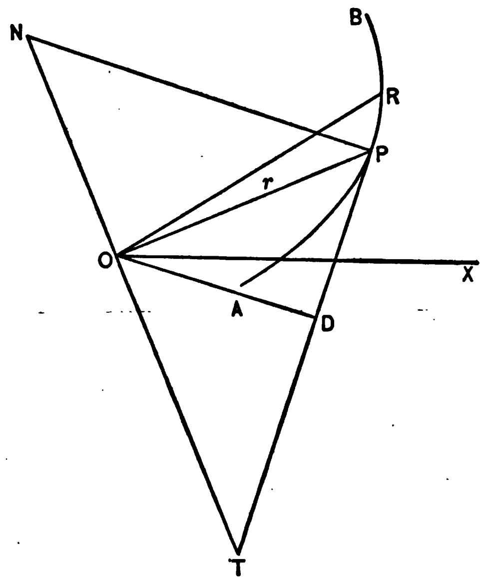

FIG. 18.

NT is drawn through the pole O, perpendicular to OP.

PT is called the *polar tangent;*

PN, the *polar normal;*

OT, the *polar subtangent;*

ON, the *polar subnormal;*

and OD, the *perpendicular on tangent* from the pole.

$$OT = \text{subtangent} = OP \tan OPT = OP\left(r\frac{d\theta}{dr}\right), \text{ by Art. 71,}$$

$$= r^2\frac{d\theta}{dr}. \qquad (1)$$

$$ON = \text{subnormal} = OP \tan OPN = OP \cot OPT$$

$$= \frac{dr}{d\theta}. \qquad (2)$$

$$PT = \text{tangent} \qquad = \sqrt{OP^2 + OT^2} = r\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}. \tag{3}$$

$$PN = \text{normal} \qquad = \sqrt{OP^2 + ON^2} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}. \tag{4}$$

$$OD = \text{perpendicular} = OP \sin OPD$$

$$= r\frac{\tan OPD}{\sqrt{1 + \tan^2 OPD}} = \frac{r^2}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}. \tag{5}$$

PROBLEMS.

1. Find the equations of the tangent and normal to the circle,

$$x^2 + y^2 = r^2.$$

By differentiation $\frac{dy}{dx} = -\frac{x}{y}$; $\therefore \frac{dy'}{dx'} = -\frac{x'}{y'}$.

Substituting this derivative in Art. 69 (1), gives

$$y - y' = -\frac{x'}{y'}(x - x'),$$

whence, $xx' + yy' = r^2$, which is the equation of the tangent.

Substituting $\frac{dy'}{dx'} = -\frac{x'}{y'}$ in Art. 69 (2), gives $y - y' = \frac{y'}{x'}(x - x')$, which is the equation of the normal.

2. Find the equations of the tangent and normal to the ellipse

$$a^2y^2 + b^2x^2 = a^2b^2.$$

3. Find the equations of the tangent and normal to the parabola

$$y^2 = 2px.$$

4. Find the equations of the tangent and normal to $y^2 = 2x^2 - x^3$ at $x = 1$. *Ans.* Equations of tangent: $y = \frac{1}{2}x + \frac{1}{2}$, $y = -\frac{1}{2}x - \frac{1}{2}$.
Equations of normal: $y = -2x + 3$, $y = 2x - 3$.

5. What is the inclination of the tangent to the curve

$$x^2y^2 = a^3(x + y), \text{ at the origin?}$$ *Ans.* 135°.

6. What is the value of the subtangent to $y = a^x$? *Ans.* m.

7. What is the value of the subnormal to $y^n = a^{n-1}x$? *Ans.* $\dfrac{y^2}{nx}$.

8. At what angle do the curves $y = \dfrac{8a^3}{x^2 + 4a^2}$ and $x^2 = 4ay$ intersect?

Ans. $71° 33' 54''$.

9. Find the values of the normal and subnormal to the cycloid

$$x = 2 \text{ arc vers } \frac{y}{2} - \sqrt{4y - y^2}, \text{ at the point where } y = 1.$$

Ans. Normal $= 2$; subnormal $= \sqrt{3}$.

10. Find the values of the tangent, normal, subtangent and subnormal to the spiral of Archimedes, whose equation is $r = a\theta$.

Ans. Subtangent $= \dfrac{r^2}{a}$, subnormal $= a$,

tangent $= r\sqrt{1 + \dfrac{r^2}{a^2}}$, normal $= \sqrt{r^2 + a^2}$.

11. Find the values of the subtangent and subnormal to $r^2 = a^2 \cos 2\theta$.

Ans. Subtangent $= -\dfrac{r^3}{a^2 \sin 2\theta}$;

subnormal $= -\dfrac{a^2}{r} \sin 2\theta$.

Art. 75. Rectilinear Asymptotes.

An asymptote to a curve is the limiting position of the tangent when the point of contact moves to an infinite distance from the origin.

Hence, any curve will have an asymptote when the point of contact of a tangent is infinitely removed from the origin, and when the tangent intersects either coördinate axis at a finite distance from the origin.

From Art. 69 (1),

$$\text{Intercept on } X = x' - y' \frac{dx'}{dy'}, \tag{1}$$

and

$$\text{intercept on } Y = y' - x' \frac{dy'}{dx'}. \tag{2}$$

If in (1) and (2) the intercept on X or Y is finite when $x' = \infty$ or $y' = \infty$, then the tangent at (x', y') is an asymptote. For example, to examine the hyperbola $a^2y^2 - b^2x^2 = - a^2b^2$ for asymptotes:

Here $$\frac{dy'}{dx'} = \frac{b^2x'}{a^2y'}.$$

Hence $$\text{intercept on } X = x' - \frac{a^2y'^2}{b^2x'}$$
$$= \frac{a^2}{x'} = 0, \text{ when } x' = \infty.$$

$$\text{Intercept on } Y = y' - \frac{b^2x'^2}{a^2y'} = - \frac{b^2}{y'} = 0, \text{ when } y' = \infty.$$

Hence, there is an asymptote passing through the origin.

$$\frac{dy'}{dx'} = \frac{b^2x'}{a^2y'} = \pm \frac{b}{a} \frac{1}{\sqrt{1 - \frac{a^2}{x^2}}} = \pm \frac{b}{a}, \text{ when } x = \infty.$$

Therefore, there are two asymptotes whose slopes are $\pm \frac{b}{a}$, and the equations of the asymptotes are $y = \pm \frac{b}{a} x$.

If, when $x' = \infty$ in (1) and (2), the intercepts on both X and Y are infinite, the curve has no asymptote corresponding to $x' = \infty$.

If when $y' = \infty$ in (1) and (2), the intercepts on X and Y are infinite, the curve has no asymptote corresponding to $y' = \infty$.

If both intercepts are zero, the asymptote passes through the origin, and its direction is found by evaluating $\frac{dy}{dx}$ for $x = \infty$.

Art. 76. Asymptotes Parallel to an Axis.

When $x = \infty$ in the equation of a curve gives a finite value of y, then there is an asymptote parallel to the X-axis. For instance, if $y = a$ when $x = \infty$ in the equation of the curve, then $y = a$ is the equation of an asymptote, because it is the equation of a straight line passing within a finite distance of the origin, and touching the curve at an infinite distance.

Likewise, when $y = \infty$ gives $x = b$ in the equation of a curve, then $x = b$ is an asymptote parallel to the Y-axis.

For example, taking the curve whose equation is $y^2 = \dfrac{x^3}{x-b}$.

Here, $y = \infty$ when $x = b$; hence $x - b = 0$ is the equation of an asymptote to the curve parallel to the Y-axis.

ART. 77. ASYMPTOTES DETERMINED BY EXPANSION.

An asymptote may sometimes be determined by solving the equation of the curve for y and expanding the second member into a series in descending powers of x.

For example, to examine $y^2 = \dfrac{x^3}{x-b}$.

Here $$y^2 = x^2\left(1 - \frac{b}{x}\right)^{-1};$$

hence $$y = \pm x\left(1 - \frac{b}{x}\right)^{-\frac{1}{2}}$$

$$= \pm x\left(1 + \frac{b}{2x} + \frac{3b^2}{8x^2} + \frac{5b^3}{16x^3} + \cdots\right). \tag{1}$$

as x approaches ∞, (1) approaches

$$y = \pm\left(x + \frac{b}{2}\right). \tag{2}$$

Hence, as x increases, the curve (1) is continually approaching the straight line (2), and (1) and (2) become tangent when $x = \infty$; therefore, $y = \pm\left(x + \dfrac{b}{2}\right)$ are the equations of two asymptotes to the curve (1).

ART. 78. ASYMPTOTES IN POLAR COÖRDINATES.

If a polar curve has an asymptote, as the point of contact is at an infinite distance from the pole, and as the tangent line passes within a finite distance from the pole, the radius vector of the point of contact is parallel to the asymptote, and the subtangent is perpendicular to the asymptote and finite. [See Fig. 18, Art. 74.]

Hence, for an asymptote, the polar subtangent $r^2\dfrac{d\theta}{dr}$ is finite for $r = \infty$. Therefore, to examine a polar curve for an asymptote, a value of θ is found which makes $r = \infty$; if the corresponding polar subtan-

gent is finite, there will be an asymptote, and if the subtangent is infinite, there is no corresponding asymptote.

For example, to examine the hyperbolic spiral $r\theta = a$ for asymptotes.

If $r = \infty$ in $r = \frac{a}{\theta}$, then $\theta = 0$.

$$\frac{d\theta}{dr} = -\frac{a}{r^2}, \text{ hence } r^2 \frac{d\theta}{dr} = -a.$$

Therefore there is an asymptote parallel to the initial line which passes at a distance a from the pole.

PROBLEMS.

Examine the following curves for asymptotes:

1. $y^3 = a^3 - x^3$. *Ans.* Asymptote, $y = -x$.

2. The circle, ellipse, and parabola. *Ans.* No asymptotes.

3. $y^3 = ax^2 + x^3$. *Ans.* $y = x + \frac{a}{3}$.

4. The cissoid, $y^2 = \frac{x^3}{2r - x}$. *Ans.* $x = 2r$.

5. $y^3 = 2ax^2 - x^3$. *Ans.* $y = -x + \frac{2}{3}a$.

6. $y = c + \frac{a^3}{(x - b)^2}$. *Ans.* $y = c$, and $x = b$.

7. The lituus, $r\theta^{\frac{1}{2}} = a$. *Ans.* The initial line.

8. $r \cos\theta = a \cos 2\theta$.

   Ans. There is an asymptote perpendicular to the initial line at a distance a to the left of the pole.

CHAPTER XII.

DIRECTION OF CURVATURE. POINTS OF INFLECTION. RADIUS OF CURVATURE. CONTACT.

Art. 79. Direction of Curvature.

A curve is *concave* towards the X-axis at any point, when in the immediate vicinity of that point it lies between the tangent and the X-axis. A curve is *convex* towards the X-axis when the tangent lies between the curve and the axis.

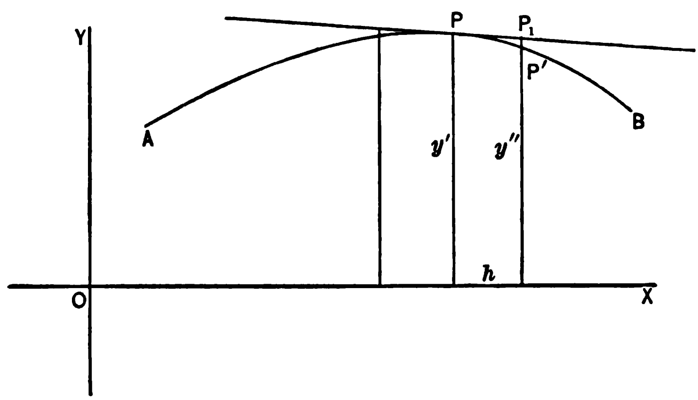

Fig. 19.

In Fig. 19, let the coördinates of P be (x', y'). The curve being concave downward, the ordinates of the curve for the abscissas $x' \pm h$, h being a very small quantity, must be less than the corresponding ordinates of the line tangent to the curve at P.

Likewise, in Fig. 20, the curve being convex downward, the ordinates of the curve for the abscissas $x' \pm h$ must be greater than the corresponding ordinates of the tangent line at P.

In either case, let $(x'+h, y'')$ be the coördinates of P'.

If $y=f(x)$ is the equation of either curve, then $y''=f(x'+h)$, and

$$y''=f(x'+h)=f(x')+f'(x')h+f''(x')\frac{h^2}{\underline{|2}}+f'''(x')\frac{h^3}{\underline{|3}}+\cdots$$

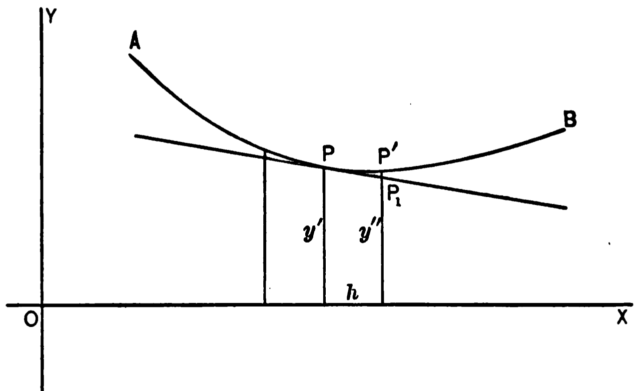

FIG. 20.

The equation of the tangent at P is

$$y-y'=f'(x')\cdot(x-x'). \tag{1}$$

If the coördinates of P_1 are $(x'+h, y_2)$,

$$y_2-y'=f'(x')[(x'+h)-x'].$$

Now $$y_2=f(x')+f'(x')h; \tag{2}$$

hence $$y''-y_2=f''(x')\frac{h^2}{\underline{|2}}+f'''(x')\frac{h^3}{\underline{|3}}+\cdots, \tag{3}$$

or if h be taken sufficiently small,

$$y''-y_2=f''(x')\frac{h^2}{\underline{|2}}. \tag{4}$$

In (4), $y''-y_2$ will have the same sign as $f''(x')$; therefore, the curve is concave to the X-axis if $f''(x')$ is negative, and convex if $f''(x')$ is positive.

If the curve is below the X-axis, y'' and y_2 are negative, and the curve is convex towards the X-axis when $-y''+y_2$ is negative, that is,

if $f''(x')$ is negative, and the curve is concave towards the X-axis when $f''(x')$ is positive.

Art. 80. Direction of Curvature in Polar Coördinates.

A curve referred to polar coördinates is *concave* to the pole at any point, when in the immediate vicinity of that point it lies between the tangent to the curve at that point and the pole. A curve is *convex* to the pole when the tangent lies between the curve and the pole.

If p is the perpendicular distance from the pole to the tangent to the curve at a point whose coördinates are (r, θ), it is evident from Fig. 21, that when the curve is concave to the pole, p increases as r increases; hence, $\frac{dp}{dr}$ is positive.

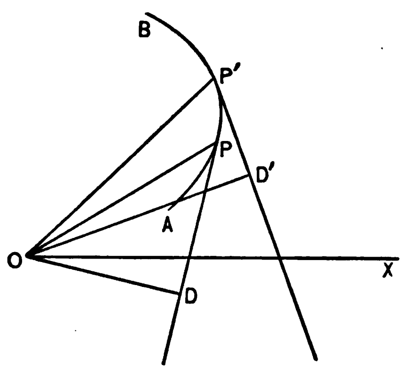

Fig. 21.

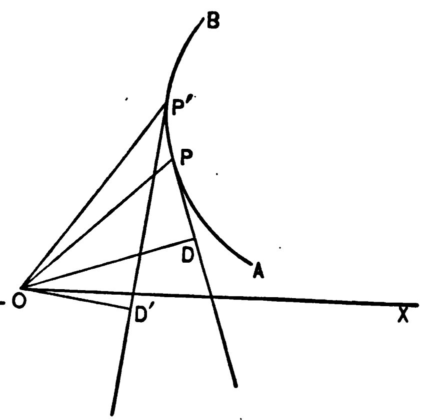

Fig. 22.

Similarly, from Fig. 22, when the curve is convex to the pole, p decreases as r increases; hence $\frac{dp}{dr}$ is negative.

If the equation of the curve is given in terms of r and θ, the equation may be transformed into an equation between r and p by aid of Art. 74 (5); then the curve is concave or convex, according as $\frac{dp}{dr}$ is positive or negative.

ART. 81. POINTS OF INFLECTION.

A point of inflection of a curve is a point where the curvature changes from concavity to convexity or the reverse. Hence, at a point of inflection the curve cuts the tangent.

By Art. 79, when $f''(x) < 0$, the curve is concave to the X-axis, and convex when $f''(x) > 0$; therefore $f''(x)$ changes sign, and hence $f''(x) = 0$ or ∞ at a point of inflection.

For example, to examine $y = \dfrac{x^3 - ax^2}{b^2}$ for points of inflection.

$$\frac{d^2y}{dx^2} = \frac{6x - 2a}{b^2} = 0; \text{ hence } x = \frac{a}{3}.$$

If
$$x < \frac{a}{3}, \text{ then } \frac{d^2y}{dx^2} < 0;$$

and if
$$x > \frac{a}{3}, \text{ then } \frac{d^2y}{dx^2} > 0.$$

Hence, $f''(x)$ changes sign at the point whose abscissa is $\dfrac{a}{3}$, and therefore this will be a point of inflection.

PROBLEMS.

1. Find the direction of curvature and point of inflection of $y = a + (x - b)^3$.

Ans. There is a point of inflection at (b, a); on the left of this point the curve is concave, while on the right it is convex.

2. Examine $y = x + 36x^2 - 2x^3 - x^4$ for points of inflection.

Ans. At $x = 2$, and $x = -3$.

3. Find points of inflection and direction of curvature of $y = \dfrac{x^3}{x^2 + 12}$.

Ans. $(-6, -\frac{9}{2})$, $(0, 0)$, $(6, \frac{9}{2})$; convex on the left of the first point, concave between first and second points, convex between second and third, and concave on the right of third point.

4. Find the direction of curvature of the lituus $r = \dfrac{a}{\theta^{\frac{1}{2}}}$.

Ans. Concave towards the pole when $r < a\sqrt{2}$;
convex towards the pole when $r > a\sqrt{2}$.

ART. 82. CURVATURE.

The total curvature of a curve between two points is the total change of direction in passing from one point to the other, and is measured by the angle formed by the tangents to the curve at the two points.

The actual curvature of a curve at a given point is the rate of change of its direction relative to that of its length.*

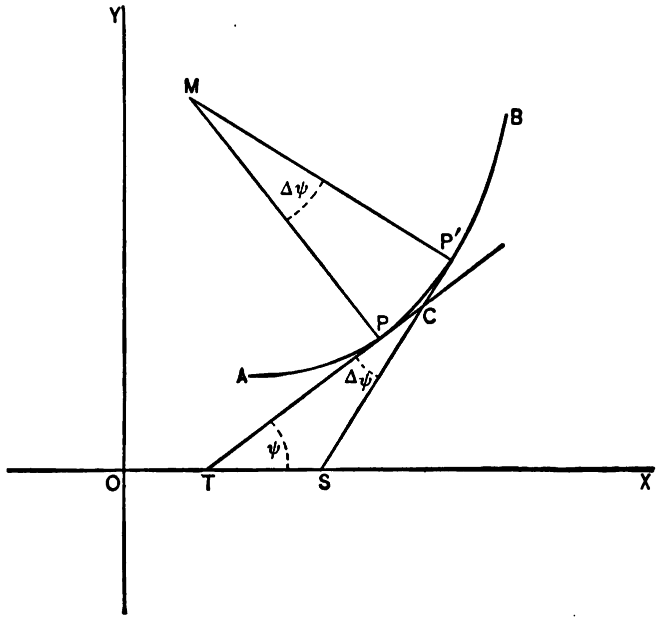

FIG. 23.

In Fig. 23, let P be any point of the curve AB. The angle $PTX = \psi$, or the angle which the tangent at P makes with the X-axis is the direction of the curve at P. Likewise the angle $P'SX$ is the

* Leibnitz defined the curvature of a curve at any point as the rate at which the curve is bending, or the rate at which the tangent is revolving per unit length of curve.

direction of the curve at P'. Then angle $TCS = \Delta\psi$ is the difference of these inclinations, and if $PP' = \Delta s$, and the point P' approaches P,

$$\text{limit}\,\frac{\Delta\psi}{\Delta s} = \frac{d\psi}{ds},$$

which is an expression for the curvature.

If the curvature is uniform; that is, if AB is the arc of a circle whose radius is r, the angle $TCS =$ angle PMP' at the centre subtended by the arc PP',

and $$\text{angle } TCS = \text{arc } \frac{PP'}{r};$$

hence $$\frac{\Delta\psi}{\Delta s} = \frac{1}{r};$$

therefore $$\frac{d\psi}{ds} = \frac{1}{r}. \tag{1}$$

Hence, the curvature of a circle is equal to the reciprocal of its radius.

Art. 83. Radius of Curvature.

The curvature of a circle varying inversely as its radius, and as any value at pleasure may be given to the radius, it follows that there is always a circle whose curvature is equal to the curvature of any curve at any point. The circle tangent to a curve at any point and having the same curvature as the curve at that point is called a *circle of curvature;* its centre is the *centre of curvature,* and its radius is the *radius of curvature* at that point.

Denoting the radius of curvature by ρ, by Art. 82 (1),

$$\rho = \frac{ds}{d\psi}. \tag{1}$$

Now it is required to find ρ in terms of x and y.

By Art. 27, $$\tan\psi = \frac{dy}{dx};$$

hence $$\sec^2\psi d\psi = \frac{d^2y}{dx};$$

therefore $$d\psi = \frac{\frac{d^2y}{dx}}{\sec^2\psi} = \frac{\frac{d^2y}{dx}}{1+\left(\frac{dy}{dx}\right)^2}. \tag{2}$$

Substituting in (1) the value of $d\psi$ just found and

$$ds = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2} \text{ from Art. 72,}$$

$$\rho = \frac{ds}{d\psi} = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}, \tag{3}$$

which is the required radius of curvature.

If l represents the length of the curve, equation (3) may be reduced to the formula $\rho = \frac{dl^3}{dx \cdot d^2y}$.

Art. 84. Radius of Curvature in Terms of Polar Coördinates.

Formula (3), Art. 83, is first transformed, any quantity t being taken as the new independent variable. The values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from (2) and (3) of Ex. 3, Art. 50, being substituted in (3) of Art. 83, putting $t = \theta$,

$$\rho = \left[\frac{\frac{dx^2}{d\theta^2}+\frac{dy^2}{d\theta^2}}{\frac{dx^2}{d\theta^2}}\right]^{\frac{3}{2}} \div \frac{\frac{d^2y}{d\theta^2}\cdot\frac{dx}{d\theta}-\frac{d^2x}{d\theta^2}\frac{dy}{d\theta}}{\frac{dx^3}{d\theta^3}},$$

$$= \frac{\left(\frac{dx^2}{d\theta^2}+\frac{dy^2}{d\theta^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{d\theta^2}\cdot\frac{dx}{d\theta}-\frac{d^2x}{d\theta^2}\cdot\frac{dy}{d\theta}}. \tag{1}$$

From the equations of transformation, $x = r\cos\theta$, and $y = r\sin\theta$, by differentiation,

$$\frac{dx}{d\theta} = -r\sin\theta + \cos\theta\frac{dr}{d\theta},$$

$$\frac{dy}{d\theta} = r\cos\theta + \sin\theta\frac{dr}{d\theta},$$

$$\frac{d^2x}{d\theta^2} = -r\cos\theta - 2\sin\theta\frac{dr}{d\theta} + \cos\theta\frac{d^2r}{d\theta^2},$$

$$\frac{d^2y}{d\theta^2} = -r\sin\theta + 2\cos\theta\frac{dr}{d\theta} + \sin\theta\frac{d^2r}{d\theta^2}.$$

Substituting these values in (1),

$$\rho=\frac{\left[r^2+\frac{dr^2}{d\theta^2}\right]^{\frac{3}{2}}}{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}}, \tag{2}$$

which is the required formula.

ART. 85. CONTACT OF DIFFERENT ORDERS.

Let $y=f(x)$ and $y=\phi(x)$ be the equations of any two curves referred to the same axes.

Giving to x a small increment h, and expanding,

$$f(x+h)=f(x)+f'(x)h+f''(x)\frac{h^2}{\underline{|2}}+f'''(x)\frac{h^3}{\underline{|3}}+\cdots, \tag{1}$$

$$\phi(x+h)=\phi(x)+\phi'(x)h+\phi''(x)\frac{h^2}{\underline{|2}}+\phi'''(x)\frac{h^3}{\underline{|3}}+\cdots. \tag{2}$$

If the two curves have a common point whose abscissa is a, then $f(a)=\phi(a)$. If, furthermore, $f'(a)=\phi'(a)$, the curves have a common tangent; this is called *contact of the first order.*

If, also, $f''(x)=\phi''(x)$, the two curves have *contact of the second order.*

In general, two curves will have contact of the nth order at $x=a$, when the following conditions are satisfied:

$$f(a)=\phi(a),\ f'(a)=\phi'(a),\ f''(a)=\phi''(a),\ \cdots f^n(a)=\phi^n(a).$$

If the curves have a common point at $x=a$, and if a be substituted for x in (1) and (2), and (2) be subtracted from (1), then

$$f(a+h)-\phi(a+h)=h[f'(a)-\phi'(a)]-\frac{h^2}{\underline{|2}}[f''(a)-\phi''(a)]$$
$$+\frac{h^3}{\underline{|3}}[f'''(a)-\phi'''(a)]+\cdots, \tag{3}$$

which is the difference between corresponding ordinates of the curves.

Now, if these curves have contact of the first order, the first term of the second member of (3) reduces to zero; if they have contact of the second order, the first two terms reduce to zero; and so on. Hence,

when the order of contact is *odd*, the first term which does not reduce to zero contains an even power of h, and the sign of the second member is the same whether h be positive or negative; therefore, one curve lies above the other on each side of their common point, and the curves do not intersect. But when the order of contact is *even*, the first term which does not vanish contains an odd power of h, and in this case the second member changes sign with h; therefore, one curve lies above the other on one side of the common point, and below it on the other side, and the curves intersect.

ART. 86. RADIUS OF THE OSCULATING CIRCLE, AND COÖRDINATES OF ITS CENTRE.

It appears from Art. 85, that $n+1$ equations must be satisfied when a curve has contact of the nth order with another curve. As an equation may be made to conform to as many different conditions as there are arbitrary constants in it, it follows that the number denoting the order of contact which any curve may have is one less than the number of arbitrary constants in its equation. The general equation of the circle has three constants; hence, at any point of a curve, the circle will have, in general, contact of the second order; this circle is called the *osculating circle*.

Let the equation of the given curve be

$$y=f(x), \tag{1}$$

and the equation of the circle

$$(x'-a)^2+(y'-b)^2=r^2. \tag{2}$$

Differentiating (2) twice,

$$x'-a+(y'-b)\frac{dy'}{dx'}=0, \tag{3}$$

and

$$1+\left(\frac{dy'}{dx'}\right)^2+(y'-b)\frac{d^2y'}{dx'^2}=0. \tag{4}$$

If (2) is the osculating circle at the point (x, y) of (1),

$$x'=x \text{ and } y'=y, \quad \frac{dy'}{dx'}=\frac{dy}{dx}, \quad \frac{d^2y'}{dx'^2}=\frac{d^2y}{dx^2}.$$

Substituting these values in (4), and solving for b:

$$b = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}. \tag{5}$$

From (3),
$$a = x - \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx}}{\frac{d^2y}{dx^2}}. \tag{6}$$

Substituting values of $(y - b)$ and $(x - a)$ from (5) and (6) in (2), after reducing,

$$r = \pm \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}. \tag{7}$$

The values of a and b in (5) and (6) are the coördinates of the centre of the osculating circle, and the value of r in (7) is its radius. Hence, by comparison with Art. 83, it will be seen that the osculating circle is the circle of curvature and the radius of the osculating circle is the radius of curvature.

Art. 87. The Osculating Circle has Contact of the Third Order where the Radius of Curvature is a Maximum or Minimum.

If ρ is to be a maximum or minimum, by Art. 66,

$$\frac{d\rho}{dx} = 0.$$

Differentiating (3) of Art. 83,

$$\frac{d\rho}{dx} = \frac{\frac{3}{2}\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} \times 2\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 - \frac{d^3y}{dx^3}\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\left(\frac{d^2y}{dx^2}\right)^2} = 0; \tag{1}$$

therefore
$$\frac{d^3y}{dx^3} = \frac{3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2}{1 + \frac{dy^2}{dx^2}}. \tag{2}$$

The same value of $\frac{d^3y}{dx^3}$ as found in (2) will be obtained by differentiating (5) of Art. 86. Therefore, the given curve and the osculating circle have the same value for $\frac{d^3y}{dx^3}$ at a point of maximum or minimum curvature; hence the contact at such a point is of the third order.

PROBLEMS.

1. Find the radius of curvature of the parabola $y^2 = 2px$ at any point, and the coördinates of the centre of curvature.

$$\frac{dy}{dx} = \frac{p}{y}, \quad \frac{d^2y}{dx^2} = -\frac{p^2}{y^3};$$

hence
$$\rho = \pm \frac{\left(1 + \frac{p^2}{y^2}\right)^{\frac{3}{2}}}{-\frac{p^2}{y^3}} = \frac{(y^2 + p^2)^{\frac{3}{2}}}{p^2} = \frac{N^3}{p^2}.$$

$$a = x - \frac{\left[1 + \frac{p^2}{y^2}\right]\frac{p}{y}}{-\frac{p^2}{y^3}} = 3x + p.$$

$$b = y + \frac{1 + \frac{p^2}{y^2}}{-\frac{p^2}{y^3}} = -\frac{y^3}{p^2}.$$

At the vertex, $x = 0$ and $y = 0$; therefore $\rho = p$, $a = p$ and $b = 0$.

2. Find the radius of curvature of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$.

Ans. $\rho = \frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4b^4}$.

3. Find the curvature of $y = x^4 - 4x^3 - 18x^2$ at the origin.

Ans. $\rho = \frac{1}{36}$.

4. Find the radius of curvature of the cycloid $x = r \operatorname{arc vers} \frac{y}{r} - \sqrt{2ry - y^2}$.

Ans. $\rho = 2\sqrt{2ry}$.

5. Find the radius of curvature of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$.

Ans. $\rho = \frac{y^2}{a}$.

6. Find the radius of curvature of the spiral of Archimedes, $r = a\theta$.

Ans. $\frac{(a^2 + r^2)^{\frac{3}{2}}}{2a^2 + r^2}$

7. Find the radius of curvature of the cardiod $r = a(1 - \cos\theta)$.

Ans. $\rho = \frac{2}{3}\sqrt{2ar}$.

8. Find the radius of curvature of the ellipse whose axes are 8 and 4, at $x = 2$, and the coördinates of the centre of curvature.

Ans. $\rho = 5.86$; $a = .38$, and $b = -3.9$.

9. Find the order of contact of

$x^3 - 3x^2 = 9y - 27$ and $3x = 28 - 9y$. *Ans.* Second order.

10. What is the order of contact of the parabola $4y = x^2 - 4$ and the circle $x^2 + y^2 - 2y = 3$? *Ans.* Third order.

11. What is the radius of curvature of the curve $16y^2 = 4x^4 - x^6$, at the points (0, 0) and (2, 0)? *Ans.* $\rho = 1$, and $\rho = 2$.

12. What is the radius of curvature of the curve $y = x^3 + 5x^2 + 6x$, at the origin? *Ans.* $\rho = 22.506$.

13. Find the radius of curvature and the coördinates of the centre of curvature of the curve $y = e^x$, at $x = 0$.

Ans. $\rho = 2\sqrt{2}$, $(a, b) = (-2, 3)$.

CHAPTER XIII.

EVOLUTES AND INVOLUTES. ENVELOPES.

ART. 88. DEFINITION OF EVOLUTE AND INVOLUTE.

The *evolute* of any curve is the curve which is the locus of the centres of all the osculating circles of the given curve; the given curve with respect to its evolute being called an *involute*.

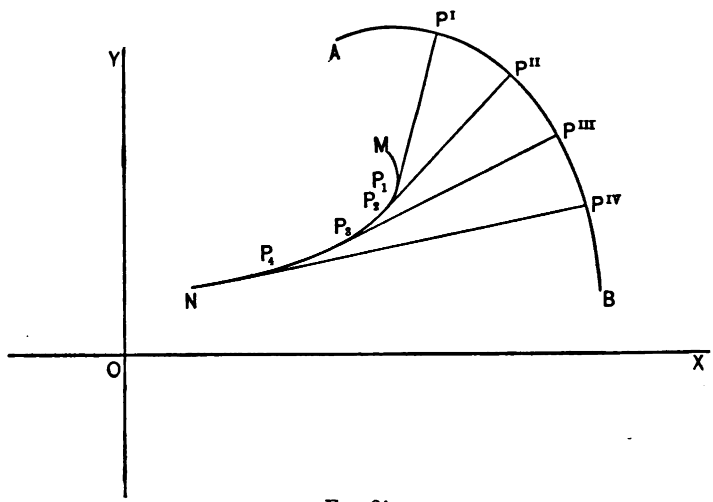

FIG. 24.

In Fig. 24, let AB be the given curve, and the centres of curvature of P', P'', P''', etc., be respectively P_1, P_2, P_3, etc.; then the curve MN, which is the locus of P_1, P_2, P_3, etc., is the evolute of AB.

ART. 89. EQUATION OF THE EVOLUTE.

The equation of the evolute is the equation which expresses the relation between the coördinates of the centres of all the osculating

circles of the involute. The values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ derived from the equation of the curve are substituted in equations (5) and (6) of Art. 86, giving two equations, which, together with the equation of the given curve, make three equations involving x, y, a and b; by combining these equations, eliminating x and y, a resulting equation will be obtained showing a relation between a and b, the coördinates of the evolute, which is the required equation.

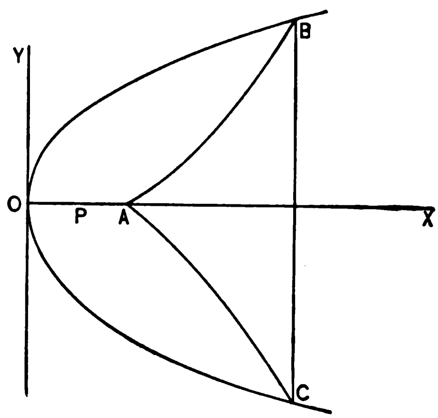

Fig. 25.

For example, to find the equation of the evolute to the common parabola, $y^2 = 2px$.

Here $$\frac{dy}{dx} = \frac{p}{y},\ \frac{d^2y}{dx^2} = -\frac{p^2}{y^3}.$$

Substituting in (5) and (6) of Art. 86:

$$b = y - \frac{y^2 + p^2}{y^2} \cdot \frac{y^3}{p^2} = -\frac{y^3}{p^2};\ \text{hence } y^2 = p^{\frac{4}{3}} b^{\frac{2}{3}}. \qquad (2)$$

$$a = x + \frac{y^2 + p^2}{y^2} \cdot \frac{p}{y} \cdot \frac{y^3}{p^2} = 3x + p;\ \text{hence } x = \frac{a - p}{3}. \qquad (3)$$

The values of y^2 and x in (2) and (3) substituted in the equation of the parabola, give $$p^{\frac{4}{3}} b^{\frac{2}{3}} = 2p\frac{a-p}{3};$$

therefore $$b^2 = \frac{8}{27p}(a - p)^3. \qquad (4)$$

Equation (4) is the equation of the evolute.

This evolute is called the semi-cubical parabola.

Constructing the evolute, its form and position is as shown in Fig. 25, where $OA = p$.

If the origin is transferred to A, the equation becomes $b^2 = \frac{8}{27p}a^3$, or as a and b are the variable coördinates, the equation may be written,

$$y^2 = \frac{8}{27p}x^3.$$

The semi-cubical parabola is so called from the nature of its equation; the equation being solved for y gives the function expressed in terms of the variable with an exponent of three halves.

Art. 90. A Normal to any Involute is Tangent to its Evolute.

Let P', in Fig. 24, be any point of the involute, whose coördinates are x' and y', and let (a, b) be the coördinates of P_1, the centre of curvature. Then the equation of the normal at P' by Art. 69 (2), is

$$y - y' = -\frac{dx'}{dy'}(x - x'). \tag{1}$$

As (1) passes through (a, b),

$$x' - a + \frac{dy'}{dx'}(y' - b) = 0. \tag{2}$$

Now if P' moves along the curve, P_1 moves along the evolute; hence a, b and y' are functions of x'.

Differentiating (2),

$$1 - \frac{da}{dx'} + \frac{d^2y'}{dx'^2}(y' - b) + \left(\frac{dy'}{dx'}\right)^2 - \frac{dy'}{dx'}\frac{db}{dx'} = 0. \tag{3}$$

But since (a, b) is on the evolute, by Art. 86 (5),

$$b = y' + \frac{1 + \left(\frac{dy'}{dx'}\right)^2}{\frac{d^2y'}{dx'^2}},$$

or,

$$(y' - b)\frac{d^2y'}{dx'^2} + 1 + \frac{dy'^2}{dx'^2} = 0. \tag{4}$$

Substituting (4) in (3),

$$-\frac{da}{dx'}=\frac{dy'}{dx'}\frac{db}{dx'}=0;\ \text{or}\ -\frac{dx'}{dy'}=\frac{db}{da}.$$

Hence, equation (2), which is the equation of the normal to the involute at (x', y'), may be written

$$y'-b=\frac{db}{da}(x'-a), \tag{5}$$

which is the equation of a tangent to the evolute at the point (a, b).

Art. 91. The Difference between Any Two Radii of Curvature of an Involute.

The equation of the circle of curvature at (x', y') is

$$(x'-a)^2+(y'-b)^2=\rho^2. \tag{1}$$

Differentiating (1), y', a, b and ρ being functions of x', gives

$$(x'-a)-(x'-a)\frac{da}{dx'}+(y'-b)\frac{dy'}{dx'}-(y'-b)\frac{db}{dx'}=\rho\frac{d\rho}{dx'}. \tag{2}$$

By Art. 90 (2),

$$x'-a+\frac{dy'}{dx'}(y'-b)=0; \tag{3}$$

and by Art. 90 (5),

$$y'-b=\frac{db}{da}(x'-a). \tag{4}$$

Combining (1) and the square of (4),

$$(x'-a)^2\left(\frac{da^2+db^2}{da^2}\right)=\rho^2. \tag{5}$$

Combining (2) and (3), and the resulting equation with (4),

$$-(x'-a)\left(\frac{da^2+db^2}{da^2}\right)=\frac{\rho d\rho}{da}. \tag{6}$$

Dividing (6) by the square root of (5), and simplifying,

$$\sqrt{da^2+db^2}=d\rho.$$

Hence, if s represents the length of the evolute, by Art. 72,

$$ds=d\rho;$$

therefore

$$\Delta s=\Delta\rho,$$

or in Fig. 24, $$P_1P_2 = P''P_2 - P'P_1,$$

or the difference between any two radii of curvature of an involute is equal to the included arc of the evolute.

Art. 92. Mechanical Construction of an Involute from its Evolute.

From the two properties of the evolute established in Arts. 90 and 91, the involute may be readily constructed from its evolute. Thus in Fig. 26, if one end of a string be fastened at N and the string be stretched along the curve NM having a pencil attached to the other

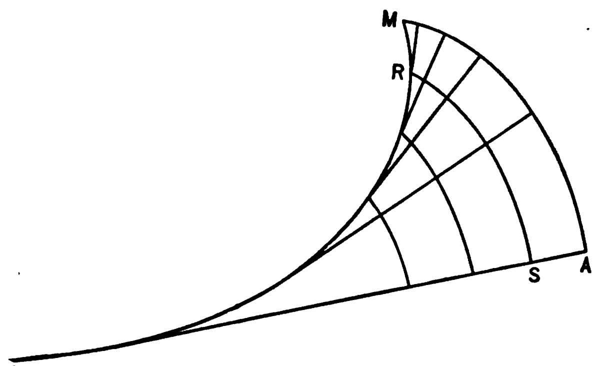

Fig. 26.

end, and then the string be gradually unwound from the evolute, always being in tension, the pencil will describe the involute MA. Every point in the string beyond N will describe an involute, as R describes RS. So while any curve can have but one evolute, as NM is the only evolute of MA, it is evident that any curve may have an infinite number of involutes. A series of curves having the same evolute are called *parallel curves.*

Art. 93. Envelopes of Curves.

If in the equation of a plane curve of the form

$$f(x, y, a) = 0,$$

different values be successively assigned to a, the several equations thus obtained will represent distinct curves, differing from each other in form and position, but belonging to the same class, or family of curves.

Now, if a is supposed to vary by infinitesimal increments, any two adjacent curves of the series will, in general, intersect, and the intersections are points of the *envelope*.

Hence, an envelope of a series of curves is the locus of the ultimate intersections of the consecutive curves.

The quantity a, which remains constant in any one curve, is called the *variable parameter*.

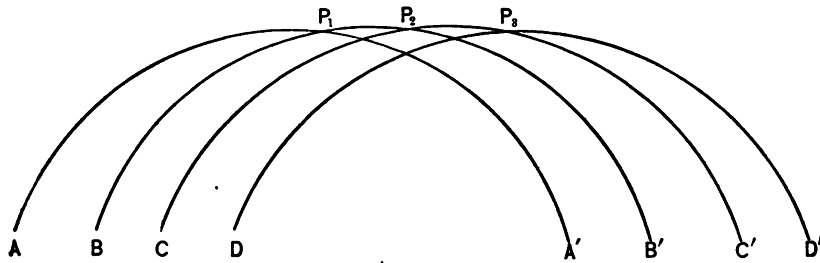

FIG. 27.

In Fig. 27, let AA', BB', etc., represent curves of a series, and a_1, a_2, etc., their respective parameters; then if $a_2 - a_1$, $a_3 - a_2$, etc., diminish indefinitely, the ultimate intersections P_1, P_2, P_3, etc., will be points of the envelope. And, at the limit, the line P_1, P_2, joins two consecutive points on the envelope and on the curve BB', and hence is tangent to both the envelope and the curve BB', then the envelope is tangent to the curve BB'.

Similarly, it may be shown that the envelope is tangent to any other curve of the series.

Hence the envelope of a family of curves is tangent to each curve of the series.

ART. 94. EQUATION OF THE ENVELOPE OF A FAMILY OF CURVES.

Let the equations of two curves of the series be

$$f(x, y, a) = 0, \tag{1}$$

and

$$f(x, y, a + \Delta a) = 0. \tag{2}$$

The coördinates of the point of intersection of (1) and (2) will satisfy both (1) and (2), and hence will also satisfy

$$f(x, y, a+\Delta a) - f(x, y, a) = 0,$$

and
$$\frac{f(x, y, a+\Delta a) - f(x, y, a)}{\Delta a} = 0. \tag{3}$$

As Δa approaches 0, the limit in equation (3) is

$$\frac{df(x, y, a)}{da} = 0. \tag{4}$$

Now the coördinates of the point of intersection of two consecutive curves satisfy both (4) and (1). Therefore, by eliminating a between (1) and (4) the resulting equation is the equation of the locus of the ultimate intersections, which is the required equation of the envelope.

For example, required the envelope of a series of curves represented by

$$y = ax - \frac{1+a^2}{4}x^2. \tag{1}$$

a being the variable parameter.

Differentiating (1) with respect to a,

$$x - \frac{ax^2}{2} = 0; \tag{2}$$

hence
$$a = \frac{2}{x}. \tag{3}$$

Combining (1) and (2), eliminating a, and reducing,

$$y = 1 - \frac{x^2}{4}, \tag{4}$$

which is the equation of the envelope.

PROBLEMS.

1 Find the equation of the evolute of the ellipse

$$A^2y^2 + B^2x^2 = A^2B^2. \tag{1}$$

Here
$$\frac{dy}{dx} = -\frac{B^2x}{A^2y}, \text{ and } \frac{d^2y}{dx^2} = -\frac{B^4}{A^2y^3};$$

hence $$a=\frac{(A^2-B^2)x^3}{A^4}, \quad \text{and} \quad x=\left(\frac{A^4a}{A^2-B^2}\right)^{\frac{1}{3}};$$

$$b=-\frac{(A^2-B^2)y^3}{B^4}, \quad \text{and} \quad y=-\left(\frac{B^4b}{A^2-B^2}\right)^{\frac{1}{3}}.$$

Substituting these values of x and y in (1),

$$(Aa)^{\frac{2}{3}}+(Bb)^{\frac{2}{3}}=(A^2-B^2)^{\frac{2}{3}},$$

which is the equation of the required evolute.

2. Find the equation of the evolute of the cycloid,

$$x=r\operatorname{vers}^{-1}\frac{y}{r}-\sqrt{2ry-y^2}.$$

Ans. $a=r\operatorname{vers}^{-1}\left(-\frac{b}{r}\right)+\sqrt{-2rb-b^2}.$

3. Find the equation of the evolute to the hypocycloid,

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=A^{\frac{2}{3}}.$$

Ans. $(a+b)^{\frac{2}{3}}+(a-b)^{\frac{2}{3}}=2A^{\frac{2}{3}}.$

4. Find the envelope of $y^2+(x-a)^2=16$, in which a is a variable parameter. *Ans.* $y=\pm 4.$

5. Find the envelope of $y=ax+\frac{m}{a}$, a being the variable parameter.

Ans. $y^2=4mx.$

6. A straight line of given length slides down between rectangular axes; required the envelope of the moving straight line.

If c represents the length of the line and a and b the intercepts, the equation is

$$\frac{x}{a}+\frac{y}{b}=1, \tag{1}$$

the relation between a and b being

$$a^2+b^2=c^2. \tag{2}$$

Differentiating (1) and (2) with respect to a and b, gives

$$-\frac{x}{a^2}\,da=\frac{y}{b^2}\,db, \tag{3}$$

and $$-ada=bdb. \tag{4}$$

Dividing (3) by (4),

$$\frac{x}{a^3}=\frac{y}{b^3}, \text{ or } \frac{\frac{x}{a}}{a^2}=\frac{\frac{y}{b}}{b^2}=\frac{\frac{x}{a}+\frac{y}{b}}{a^2+b^2}=\frac{1}{c^2};$$

hence $$a=(xc^2)^{\frac{1}{3}},$$

and $$b=(yc^2)^{\frac{1}{3}},$$

which substituted in (2), gives

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}},$$

which is the equation of the hypocycloid.

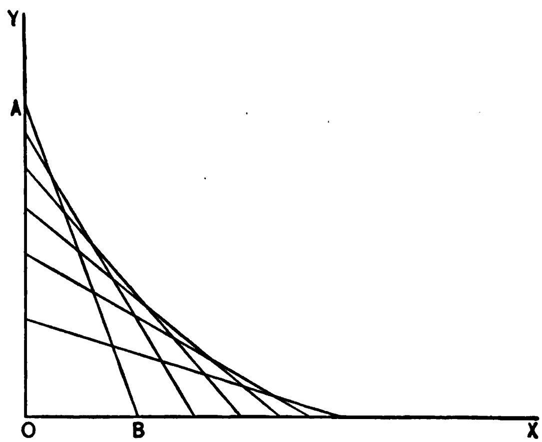

FIG. 28.

7. Find the envelope of a series of concentric ellipses, the area and direction of axes being constant.

Ans. If $c=$ area, the equation of the envelope is $xy=\pm\frac{c}{2\pi}$.

8. Find the envelope of $x\cos a+y\sin a=p$, in which a is the variable parameter. *Ans.* $x^2+y^2=p^2$.

CHAPTER XIV.

SINGULAR POINTS.

ART. 95. DEFINITIONS.

A *singular point* is a point of a curve which has some peculiarity not common to other points of the curve, and not depending on the position of the coördinate axes.

The most important singular points are:

1st. Points of maximum and minimum ordinates; 2d. Points of inflection; 3d. Multiple points; 4th. Cusps; 5th. Conjugate points; 6th. Stop points; 7th. Shooting points.

Points of maximum and minimum ordinates have been considered in Chapter X., and points of inflection in Art. 81.

ART. 96. MULTIPLE POINTS.

A *multiple point* is a point common to two or more branches of a curve.

There are two species of multiple points: 1st. Points of multiple intersection, or where two or more branches of a curve intersect; 2d. Points of osculation, or where two or more branches are tangent to each other.

Multiple points are double, triple, etc., as two, three, or more branches meet at the same point.

At a multiple point there will be as many tangents, and therefore as many values of $\frac{dy}{dx}$ as there are branches. If the values of $\frac{dy}{dx}$ are unequal, the multiple point will be one of the first species, but if the values of $\frac{dy}{dx}$ are equal, it will be one of the second species.

Let $$u = f(x, y) = 0 \tag{1}$$

be the equation of the curve freed of radicals.

Then, by Art. 47, $$\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}.$$

And since differentiation never introduces radicals when the function contains none, the value of $\frac{dy}{dx}$ cannot contain radicals, and therefore cannot have more than one value unless it assumes the form $\frac{0}{0}$. Hence the condition for a multiple point is $\frac{dy}{dx} = \frac{0}{0}$.

Therefore, to examine for multiple points, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ as obtained from the equation of the curve are placed equal to zero, and the corresponding values of x and y are found. If these values of x and y are real and satisfy (1), they may determine multiple points. Then $\frac{dy}{dx} = \frac{0}{0}$ is evaluated for the critical values of x and y, and every real value determines one branch passing through the multiple point.

PROBLEMS.

1. Examine the curve $y^2 - (x - a)^2 x = 0$ for a multiple point.

Here $$\frac{\partial u}{\partial x} = -2(x - a)x - (x - a)^2 = 0; \tag{1}$$

and $$\frac{\partial u}{\partial y} = 2y = 0. \tag{2}$$

Solving (1) and (2) for x and y, gives

$$\begin{Bmatrix} x = a \\ y = 0 \end{Bmatrix}, \text{ and } \begin{Bmatrix} x = \frac{a}{3} \\ y = 0 \end{Bmatrix}.$$

But only the first point is to be examined, as the second point does not satisfy the equation of the curve.

$$\frac{dy}{dx} = -\frac{-2(x-a)x-(x-a)^2}{2y} = \pm\frac{3x-a}{2\sqrt{x}}$$
$$= \pm\sqrt{a}, \text{ when } x = a.$$

Therefore the multiple point is a double point of the first kind, as shown in Fig. 29.

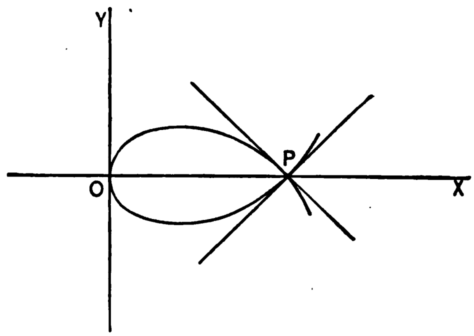

FIG. 29.

2. Examine the curve $x^4 + 2ax^2y - ay^3 = 0$, for multiple points.

$$\frac{\partial u}{\partial x} = 4x^3 + 4axy = 0; \tag{1}$$

$$\frac{\partial u}{\partial y} = 2ax^2 - 3ay^2 = 0. \tag{2}$$

Combining (1) and (2) gives three pairs of values for x and y, but the only pair that satisfies the equation of the curve is $(0, 0)$.

$$\frac{dy}{dx} = \frac{4x^3 + 4axy}{3ay^2 - 2ax^2} = \frac{0}{0}, \text{ when } \begin{cases} x = 0 \\ y = 0. \end{cases}$$

Evaluating by Art. 59, and representing $\frac{dy}{dx}$ by p and $\frac{dp}{dx}$ by p',

$$\frac{dy}{dx} = p = \frac{12x^2 + 4ay + 4axp}{6ayp - 4ax} = \frac{0}{0}, \text{ when } \begin{cases} x = 0 \\ y = 0, \end{cases}$$

$$= \frac{24x + 8ap + 4axp'}{6ap^2 + 6ayp' - 4a} = \frac{8ap}{6ap^2 - 4a}, \text{ when } \begin{cases} x = 0 \\ y = 0. \end{cases}$$

Hence $$p(6ap^2 - 4a) = 8ap;$$

and $$p = \frac{dy}{dx} = 0, \ +\sqrt{2}, \text{ and } -\sqrt{2}.$$

Therefore there is a triple point of the first kind at the origin.

3. Examine $y^2 = a^2x^2 - x^4$ for a multiple point.

Ans. **There is a double point of the first kind at the origin where**

$$\frac{dy}{dx} = \pm a.$$

4. Show that the curve $y^2 = x^3 + x^4$ has a point of osculation at the origin.

Art. 97. Cusps.

A *cusp* is a point at which two branches of a curve are tangent to each other and terminate.

Cusps are therefore multiple points of the second species.

There are two kinds of cusps: 1st. When the two branches lie on opposite sides of the common tangent; 2d. When the two branches are on the same side of the common tangent.

Since a cusp is a particular kind of multiple point, curves are examined for cusps as for multiple points. But as a cusp is distinguished from a multiple point by both branches stopping at the point, the curve must be traced in the vicinity of the point in question to determine a cusp. If the two values of $\frac{d^2y}{dx^2}$ at the cusp have contrary signs, the cusp is of the first kind, and if they have the same sign, the cusp is of the second species.

The vertex of the semi-cubical parabola is a cusp of the first kind. [See point A, Fig. 25.]

The curve $(y - x^2)^2 = x^5$ has a cusp of the second species, determined as follows:

Taking the square root of each member of the equation,

$$y = x^2 \pm x^{\frac{5}{2}}; \tag{1}$$

hence

$$\frac{dy}{dx} = 2x \pm \tfrac{5}{2} x^{\frac{3}{2}}, \tag{2}$$

and

$$\frac{d^2y}{dx^2} = 2 \pm \tfrac{15}{4} x^{\frac{1}{2}}. \tag{3}$$

In (1), if $x = 0$, then $y = 0$; if x is negative, y is imaginary; if x is positive, y has two real values. Hence, the curve has two branches on

the right of the Y-axis which meet and terminate at the origin. The locus of the equation is shown in Fig. 30.

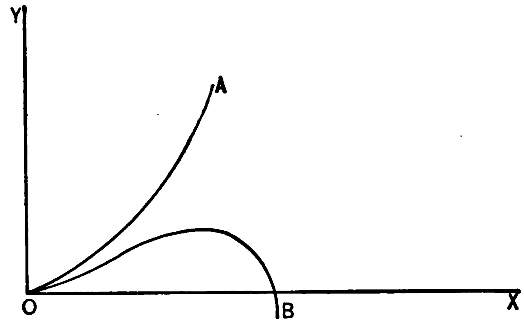

FIG. 30.

In (2) $\frac{dy}{dx} = 0$, when $x = 0$; hence the X-axis is tangent to both branches, and there is a cusp at the origin.

In (3), when a value slightly greater than 0 is substituted for x, the two values of $\frac{d^2y}{dx^2}$ are both positive; hence the cusp is of the second species.

98. Conjugate Points. Stop Points. Shooting Points.

A *conjugate* or *isolated point* is a point whose coördinates satisfy the equation of a curve, but through which the curve does not pass. As the conjugate point is detached from the curve, if the substitutions of $a + b$ and $a - b$ for x in the equation of the curve, b being very small, give imaginary values for y, then there is a conjugate point whose abscissa is a.

Or, if at any point whose coördinates satisfy the equation of a curve, $\frac{dy}{dx}$ is imaginary, this point will be a point through which no branches pass, and hence will be a conjugate point.

For example, to examine $y^2 = (x - 1)^2(x - 2)$ for conjugate points.

The point (1, 0) will be such a point, for if some value a little greater or a little less than 1 be substituted for x in the equation, the

resulting value of y will be imaginary, yet the point (1, 0) satisfies the equation. Or by the second method:

$$\frac{dy}{dx}=\frac{3x-5}{2\sqrt{x-2}}.$$

Now the point (1, 0) which satisfies the equation of the curve makes $\frac{dy}{dx}$ imaginary, and hence is a conjugate point.

In Fig. 31, MN is the curve and P is the conjugate point.

A *stop point* is a point of a curve at which a branch suddenly ends. For example, to examine $y=x\log x$ for a stop point. Here, for any positive value of x, y has one real value; when $x=0, y=0$; when x is negative, y is imaginary; therefore the origin is a stop point.

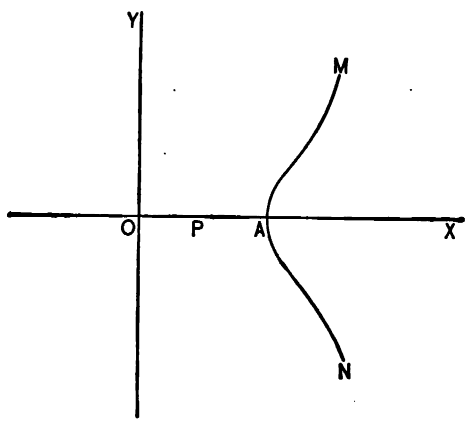

FIG. 31.

A *shooting point* is a point of a curve at which two or more branches terminate without having a common tangent.

For example, to examine $y=x\tan^{-1}\frac{1}{x}$ for shooting points.

Here, $$\frac{dy}{dx}=\tan^{-1}\frac{1}{x}-\frac{x}{1+x^2}.$$

When $x=0$, then $y=0$, and $\frac{dy}{dx}=\pm\frac{\pi}{2}$.

If x be positive and approach zero as its limit, ultimately $y=0$ and $\frac{dy}{dx}=\frac{\pi}{2}$; but if x be negative, ultimately $y=0$ and $\frac{dy}{dx}=-\frac{\pi}{2}$. Hence two branches meet at the origin, one inclined $\tan^{-1}\left(\frac{\pi}{2}\right)$ and the other inclined $\tan^{-1}\left(-\frac{\pi}{2}\right)$. Therefore the origin is a shooting point.

Stop points and shooting points occur only in transcendental curves, and may be discovered in any curve by tracing the curve in the vicinity of the singular points.

CHAPTER XV.

INTEGRATION OF RATIONAL FRACTIONS.

ART. 99. RATIONAL FRACTIONS.

A *rational fraction* is one whose numerator and denominator are rational. If the degree of the numerator is equal to or greater than the degree of the denominator, the fraction can be reduced by division to the sum of several integral terms and a fraction whose numerator is of a lower degree than its denominator. For example,

$$\frac{x^4 + 3x^3}{x^2 + 2x + 1}dx = x^2dx + x\,dx - 3\,dx + \frac{5x + 3}{x^2 + 2x + 1}dx,$$

in which the last term is the only fractional term. So it is necessary to consider only rational fractions in which the degree of the numerator is less than the degree of the denominator.

A rational fraction is integrated by decomposing it into a number of simpler partial fractions, which can be integrated separately.

CASE 1. When the denominator can be resolved into n real and unequal factors of the first degree.

Let $\dfrac{f(x)}{\phi(x)}dx$ represent a rational fraction, whose denominator may be resolved into the factors $(x-a)$, $(x-b)$, $\cdots$ $(x-l)$, real, unequal and of the first degree.

Assume
$$\frac{f(x)}{\phi(x)} \equiv \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \cdots \frac{L}{x-l}, \tag{1}$$

in which $A, B, C, \cdots L$ are undetermined coefficients. Clearing (1) of fractions,

$$f(x) \equiv A(x-b)(x-c)\cdots(x-l) + B(x-a)(x-c)\cdots(x-l) + \cdots \\ + L(x-a)(x-b)\cdots(x-k). \tag{2}$$

Performing the indicated operations in (2) and equating the coefficients of like powers of x in the two members by the Principle of

Undetermined Coefficients, will give n equations from which A, B, C, etc., may be obtained.

Or since (2) is true for all values of x, a may be substituted for x, which gives

$$A = \frac{f(a)}{(a-b)(a-c)\cdots(a-l)}. \tag{3}$$

By substituting b for x, the value of B is obtained, and so on; finally when l is substituted for x, it follows that

$$L = \frac{f(l)}{(l-a)(l-b)\cdots(l-k)}. \tag{4}$$

These values of A, B, C, etc., are substituted in (1), dx is introduced as a factor in each term, and each term is then integrated.

PROBLEMS.

1. Find $\int \frac{(x^2+x-1)}{x^3+x^2-6x}dx.$

$$x^3+x^2-6x = x(x+3)(x-2).$$

Assume
$$\frac{x^2+x-1}{x^3+x^2-6x} \equiv \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-2}. \tag{1}$$

Therefore $x^2+x-1 \equiv A(x+3)(x-2) + Bx(x-2) + Cx(x+3)$.

Substituting $x=0$, gives $-1=-6A$; hence $A=\frac{1}{6}$.

Substituting $x=-3$, gives $5=15B$; hence $B=\frac{1}{3}$.

Substituting $x=2$, gives $5=10C$; hence $C=\frac{1}{2}$.

Substituting these values of A, B and C in (1), introducing dx, and taking the integral of each member,

$$\int \frac{x^2+x-1}{x^3+x^2-6x}dx = \tfrac{1}{6}\int\frac{dx}{x} + \tfrac{1}{3}\int\frac{dx}{x+3} + \tfrac{1}{2}\int\frac{dx}{x-2}$$
$$= \tfrac{1}{6}\log x + \tfrac{1}{3}\log(x+3) + \tfrac{1}{2}\log(x-2)$$
$$= \log[x^{\frac{1}{6}}(x+3)^{\frac{1}{3}}(x-2)^{\frac{1}{2}}].$$

2. $\int \frac{(5x+1)dx}{x^2+x-2} = \log[(x-1)^2(x+2)^3].$

3. $\displaystyle\int \frac{(x^2+2)\,dx}{x^4-5x^2+4} = \tfrac{1}{2}\log\frac{x^2-x-2}{x^2+x-2}.$

4. $\displaystyle\int \frac{dx}{a^2-b^2x^2} = \frac{1}{2ab}\log\left(\frac{a+bx}{a-bx}\right).$

5. $\displaystyle\int \frac{(2x+3)\,dx}{x^3+x^2-2x} = \log\frac{(x-1)^{\frac{5}{3}}}{x^{\frac{3}{2}}(x+2)^{\frac{1}{6}}}.$

6. $\displaystyle\int \frac{(2+3x-4x^2)\,dx}{4x-x^3} = \log\left[x^{\frac{1}{2}}(2+x)^{\frac{5}{2}}(2-x)\right].$

Case 2. When the denominator can be resolved into n real and equal factors of the first degree.

Let $\dfrac{f(x)}{\phi(x)}\,dx$ represent a rational fraction whose denominator can be resolved into n factors each equal to $x-a$. In this case the method of decomposition of the preceding case is not applicable. Take, for example, $\dfrac{2x^2+x}{(x-a)^3}$.

Forming the partial fractions as before would give

$$\frac{2x^2+x}{(x-a)^3} = \frac{A}{x-a} + \frac{B}{x-a} + \frac{C}{x-a}. \tag{1}$$

But if the fractions in the second member are added,

$$\frac{2x^2+x}{(x-a)^3} = \frac{A+B+C}{x-a}, \tag{2}$$

in which $A+B+C$ must be regarded as a single constant, and evidently (2) cannot be an identical equation, as this would give three independent equations containing but one quantity, $A+B+C$, to be determined.

The partial fractions are assumed as follows:

$$\frac{f(x)}{\phi(x)} \equiv \frac{A}{(x-a)^n} + \frac{B}{(x-a)^{n-1}} + \frac{C}{(x-a)^{n-2}} + \cdots \frac{L}{(x-a)}. \tag{1}$$

Clearing (1) of fractions,

$$f(x) \equiv A + B(x-a) + C(x-a)^2 + \cdots L(x-a)^{n-1}. \tag{2}$$

The values of A, B, C, etc., in (2), are found by the Principle of Undetermined Coefficients, then substituted in (1), after which dx is introduced, and each term is integrated separately.

PROBLEMS.

1. Find $\int \frac{(x^2+1)}{(x-1)^3}dx.$

Assume $\qquad \frac{x^2+1}{(x-1)^3} \equiv \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)}.$

Hence $\qquad x^2+1 \equiv A + B(x-1) + C(x-1)^2$

$$\equiv A + Bx - B + Cx^2 - 2Cx + C.$$

Therefore $\qquad C = 1,\ B - 2C = 0, \text{ and } A - B + C = 1;$

whence $\qquad C = 1,\ B = 2, \text{ and } A = 2.$

Therefore $\int \frac{(x^2+1)}{(x-1)^3}dx = \int \frac{2\,dx}{(x-1)^3} + \int \frac{2\,dx}{(x-1)^2} + \int \frac{dx}{(x-1)}$

$$= -\frac{1}{(x-1)^2} - \frac{2}{x-1} + \log(x-1).$$

2. $\int \frac{(3x^2-2)dx}{(x+2)^3} = \frac{12x+19}{(x+2)^2} + 3\log(x+2).$

When the denominator of a rational fraction may be resolved into both equal and unequal factors of the first degree, the two methods must be combined.

3. $\int \frac{x^2dx}{(x+2)^2(x+1)} = \frac{4}{x+2} + \log(x+1).$

4. $\int \frac{x^2dx}{(x+2)^2(x+4)^2} = -\frac{5x+12}{x^2+6x+8} + \log\left(\frac{x+4}{x+2}\right)^2.$

5. $\int \frac{x^2-4x+3}{x^3-6x^2+9x}dx = \log\left[x(x-3)^2\right]^{\frac{1}{3}}.$

6. $\int \frac{dx}{(x^2-2)^2} = -\frac{x}{4(x^2-2)} + \frac{1}{8\sqrt{2}}\log\frac{x+\sqrt{2}}{x-\sqrt{2}}.$

CASE 3. When some of the simple factors of the denominator are imaginary.

As the denominator is real, the imaginary factors must occur in pairs, and of the forms

$$x \pm a + b\sqrt{-1}, \text{ and } x \pm a - b\sqrt{-1},$$

whose product is the real quadratic factor

$$(x \pm a)^2 + b^2. \tag{1}$$

For a single quadratic factor such as (1), the corresponding partial fraction will have the form $\dfrac{Ax+b}{(x \pm a)^2 + b^2}$, because each fraction of this form increases by two the degree of the equation when it is cleared of fractions and therefore increases by two the number of the equations for determining A, B, C, etc.; hence its numerator should add two to the number of these undetermined constants.

If the denominator contains n equal quadratic factors, being of the form

$$[(x \pm a)^2 + b^2]^n,$$

the partial fractions may be assumed as follows:

$$\frac{f(x)}{\phi(x)} \equiv \frac{Ax+B}{[(x \pm a)^2 + b^2]^n} + \frac{Cx+D}{[(x \pm a)^2 + b^2]^{n-1}} \cdots \frac{Kx+L}{(x \pm a)^2 + b^2}. \tag{2}$$

The values of A, B, C, etc., are determined from (2), as in the preceding cases.

PROBLEMS.

1. $\displaystyle\int \frac{x^2dx}{x^4 + x^2 - 2}$.

Assume $\quad \displaystyle\frac{x^2}{x^4+x^2-2} \equiv \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+2}.$

Hence $\quad A = -\frac{1}{6}, \quad B = \frac{1}{6}, \quad C = 0, \quad D = \frac{2}{3}.$

$$\int \frac{x^2dx}{x^4+x^2-2} = -\frac{1}{6}\int \frac{dx}{x+1} + \frac{1}{6}\int \frac{dx}{x-1} + \frac{2}{3}\int \frac{dx}{x^2+2}$$

$$= \tfrac{1}{6} \log \frac{x-1}{x+1} + \frac{\sqrt{2}}{3} \tan^{-1} \frac{x}{\sqrt{2}}.$$

2. $\displaystyle\int\frac{x\,dx}{(x+1)(x^2+1)}=\tfrac{1}{2}\text{ arc tan } x+\log\frac{(x^2+1)^{\frac{1}{4}}}{(x+1)^{\frac{1}{2}}}.$

3. $\displaystyle\int\frac{2\,x\,dx}{(x^2+1)(x^2+3)}=\log\left(\frac{x^2+1}{x^2+3}\right)^{\frac{1}{2}}.$

4. $$\int\frac{dx}{(x-1)^2(x^2+1)^2}=-\frac{1}{4\,(x-1)}-\tfrac{1}{2}\log(x-1)+\tfrac{1}{4}\tan^{-1}x$$
$$-\frac{1}{4\,(x^2+1)}+\tfrac{1}{4}\log(x^2+1).$$

5. $\displaystyle\int\frac{(x^3-1)\,dx}{x^2-4}=\frac{x^2}{2}+\log\left[(x+2)^{\frac{9}{4}}(x-2)^{\frac{7}{4}}\right].$

6. $\displaystyle\int\frac{dx}{(x^2+1)(x^2+x+1)}=\tfrac{1}{2}\log\frac{x^2+x+1}{x^2+1}+\frac{1}{\sqrt{3}}\text{ arc tan}\frac{2\,x+1}{\sqrt{3}}.$

CHAPTER XVI.

INTEGRATION OF IRRATIONAL FUNCTIONS.

ART. 100. IRRATIONAL FUNCTIONS.

Very few irrational functions are integrable. When an irrational function cannot be directly integrated by one of the elementary formulas, an effort is made to transform it into an equivalent rational function of another variable by making suitable substitutions. When the rationalization can be effected, the integral may be found.

ART. 101. IRRATIONAL FUNCTIONS CONTAINING ONLY MONOMIAL SURDS.

An irrational function containing only monomial surds may be rationalized by substituting for the variable a new variable with an exponent equal to the least common multiple of the denominators of the fractional exponents in the function.

For example, to find $\displaystyle\int\frac{x^{\frac{1}{2}}-2\,x^{\frac{1}{3}}}{1+x^{\frac{1}{3}}}\,dx.$

Assume $\qquad x=z^6$; then $x^{\frac{1}{2}}=z^3,\ x^{\frac{1}{3}}=z^2$, and $dx=6\,z^5dz.$

Hence

$$\begin{aligned}\int\frac{x^{\frac{1}{2}}-2\,x^{\frac{1}{3}}}{1+x^{\frac{1}{3}}}\,dx&=6\int\frac{z^3-2\,z^2}{1+z^2}\,z^5dz=6\int\frac{z^8-2\,z^7}{1+z^2}\,dz\\&=6\int\left(z^6-2\,z^5-z^4+2\,z^3+z^2-2\,z-1+\frac{2\,z+1}{z^2+1}\right)dz\\&=\frac{6\,z^7}{7}-2\,z^6-\tfrac{6}{5}\,z^5+3\,z^4+2\,z^3-6\,z^2-6\,z\\&\qquad+6\log(z^2+1)+6\operatorname{arc}\tan z\\&=\frac{6\,x^{\frac{7}{6}}}{7}-2\,x-\tfrac{6}{5}\,x^{\frac{5}{6}}+3\,x^{\frac{2}{3}}+2\,x^{\frac{1}{2}}-6\,x^{\frac{1}{3}}-6\,x^{\frac{1}{6}}\\&\qquad+6\log(x^{\frac{1}{3}}+1)+6\operatorname{arc}\tan x^{\frac{1}{6}}.\end{aligned}$$

PROBLEMS.

1. $\int \frac{2x^{\frac{1}{2}} - 3x^{\frac{2}{3}}}{5x^{\frac{1}{6}}} dx = \frac{3}{10} x^{\frac{4}{3}} - \frac{2}{5} x^{\frac{3}{2}}.$

2. $\int \frac{dx}{x^{\frac{7}{6}} + x^{\frac{4}{3}}} = -\frac{6}{x^{\frac{1}{6}}} + \log \frac{(x^{\frac{1}{3}} + 1)^6}{x}.$

3. $\int \frac{3x^{\frac{1}{2}} dx}{2x^{\frac{1}{2}} - x^{\frac{2}{3}}} = -18 \left[\frac{x^{\frac{5}{6}}}{5} + \frac{x^{\frac{2}{3}}}{2} + \frac{4x^{\frac{1}{2}}}{3} + 4x^{\frac{1}{3}} + 16x^{\frac{1}{6}} + 32 \log (2 - x^{\frac{1}{6}}) \right].$

4. $\int \frac{x^{\frac{1}{6}} + 1}{x^{\frac{7}{6}} + x^{\frac{5}{4}}} dx = -\frac{6}{x^{\frac{1}{6}}} + \frac{12}{x^{\frac{1}{12}}} + 2 \log x - 24 \log (x^{\frac{1}{12}} + 1).$

5. $\int \frac{2x^{\frac{1}{2}} - 3x^{\frac{1}{6}}}{3x^{\frac{2}{3}} + x^{\frac{3}{4}}} dx = 12 \left(\frac{2}{9} x^{\frac{3}{4}} - \frac{3}{4} x^{\frac{2}{3}} + \frac{18}{7} x^{\frac{7}{12}} - 9x^{\frac{1}{2}} \right)$

$+ 1908 \left[\frac{1}{5} x^{\frac{5}{12}} - \frac{3}{4} x^{\frac{1}{3}} + 3x^{\frac{1}{4}} - \frac{27}{2} x^{\frac{1}{6}} + 81 x^{\frac{1}{12}} - 243 \log (x^{\frac{1}{12}} + 3) \right].$

ART. 102. FUNCTIONS CONTAINING ONLY BINOMIAL SURDS OF THE FIRST DEGREE.

A function which involves no surd except one of the form $(a + bx)^{\frac{m}{n}}$ can be rationalized by assuming $a + bx = z^n$, as follows:

Let $f(x, \sqrt[n]{a + bx})$ be the function.

Assume $$z = \sqrt[n]{a + bx};$$

then $$z^n = a + bx,$$

$$nz^{n-1} dz = b\, dx,$$

$$dx = \frac{nz^{n-1} dz}{b}.$$

And $$x = \frac{z^n - a}{b}.$$

Therefore $$\int f(x, \sqrt[n]{a + bx})\, dx = \frac{n}{b} \int f\left(\frac{z^n - a}{b}, z\right) z^{n-1} dz,$$

which is rational, and therefore can be integrated.

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PROBLEMS.

1. $\int \frac{x\,dx}{\sqrt{1+x}}$.

Assume $1+x=z^2$. Then $x=z^2-1$ and $dx=2z\,dz$.

Hence $$\int\frac{x\,dx}{\sqrt{1+x}}=\int\frac{(z^2-1)2z\,dz}{z}=2\int(z^2dz-dz)$$
$$=\tfrac{2}{3}z^3-2z$$
$$=\tfrac{2}{3}(1+x)^{\frac{3}{2}}-2(1+x)^{\frac{1}{2}}.$$

2. $\int\frac{dx}{x\sqrt{1+x}}=\log\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}$.

3. $\int(x+\sqrt{x+2}+\sqrt[3]{x+2})\,dx$
$$=\tfrac{1}{2}(x+2)^2-2(x+2)+\tfrac{2}{3}(x+2)^{\frac{3}{2}}+\tfrac{3}{4}(x+2)^{\frac{4}{3}}.$$

4. $\int\frac{\sqrt{1+x}\,dx}{\sqrt{x-1}}=\log(x+\sqrt{x^2-1})+\sqrt{x^2-1}$.

5. $\int\frac{y\,dy}{(2r-y)^{\frac{1}{2}}}=-\tfrac{2}{3}(4r+y)(2r-y)^{\frac{1}{2}}$.

6. $\int\frac{x^2dx}{(4x+1)^{\frac{5}{2}}}=\frac{6x^2+6x+1}{12(4x+1)^{\frac{3}{2}}}$.

7. $\int\frac{dx}{\sqrt[4]{1+\sqrt{1-x}}}=-8[\tfrac{1}{7}(1+\sqrt{1-x})^{\frac{7}{4}}-\tfrac{1}{3}(1+\sqrt{1-x})^{\frac{3}{4}}]$.

Assume $z=\sqrt[4]{1+\sqrt{1-x}}$.

ART. 103. FUNCTIONS HAVING THE FORM $\frac{x^{2n+1}dx}{(a+bx^2)^{\frac{m}{2}}}$, IN WHICH n IS A POSITIVE INTEGER.

Expressions of this form may be integrated as in Art. 102.

For example, find $\int\frac{x^3dx}{\sqrt{1-x^2}}$.

Assume $1-x^2=z^2$; then $x^2=1-z^2$, and $x\,dx=-z\,dz$.

Therefore $\displaystyle\int \frac{x^2 \cdot x\,dx}{\sqrt{1-x^2}} = -\int \frac{(1-z^2)z\,dz}{z} = -\int (1-z^2)\,dz$

$$= -(z - \tfrac{1}{3}z^3) = \tfrac{1}{3}(1-x^2)^{\frac{3}{2}} - (1-x^2)^{\frac{1}{2}}.$$

ART. 104. FUNCTIONS HAVING THE FORM $f\left(x, \sqrt[n]{\dfrac{ax+b}{cx+d}}\right)dx.$

In this form the assumption may be made

that $$z = \sqrt[n]{\frac{ax+b}{cx+d}};$$

then $$z^n = \frac{ax+b}{cx+d},$$

and $$x = \frac{b-dz^n}{cz^n-a};$$

therefore $$dx = \frac{n(ad-bc)z^{n-1}}{(cz^n-a)^2}\,dz.$$

The substitution of these values will make the function rational.

PROBLEMS.

1. $\displaystyle\int \frac{x^3dx}{(1+x^2)^{\frac{5}{2}}} = -\frac{3x^2+2}{3(1+x^2)^{\frac{3}{2}}}.$

2. $\displaystyle\int \frac{x^5dx}{\sqrt{2x^2+1}} = \frac{3x^4-2x^2+2}{30}\sqrt{2x^2+1}.$

3. $\displaystyle\int \frac{x^5dx}{(2+3x^2)^{\frac{5}{2}}} = -\frac{(4+9x^2)}{27(2+3x^2)^{\frac{3}{2}}}.$

4. $\displaystyle\int \frac{x\,dx}{x^2+2\sqrt{3-x^2}} = \tfrac{1}{4}\log(\sqrt{3-x^2}+1) + \tfrac{3}{4}\log(\sqrt{3-x^2}-3).$

5. $\displaystyle\int \sqrt[3]{\frac{1-x}{1+x}}\,\frac{dx}{(1+x)^2} = -\frac{3}{8}\sqrt[3]{\left(\frac{1-x}{1+x}\right)^4}.$

Art. 106. Functions containing only Trinomial Surds of the Form $\sqrt{a+bx+cx^2}$.

Case 1. When c is positive.

After factoring out $\sqrt{c}$, the surd may be written $\sqrt{A+Bx+x^2}$.

Assume $$\sqrt{A+Bx+x^2}=z-x;$$

then $$A+Bx=z^2-2zx,\quad x=\frac{z^2-A}{B+2z},$$

and $$dx=\frac{2(z^2+Bz+A)\,dz}{(B+2z)^2}.$$

Therefore $$\sqrt{A+Bx+x^2}=z-\frac{z^2-A}{B+2z}=\frac{z^2+Bz+A}{2z+B}.$$

Thus the given function may be transformed into an equivalent rational function of another variable.

Case 2. When c is negative.

After taking out the factor $\sqrt{c}$, the surd may be written

$$\sqrt{A+Bx-x^2}.$$

Assume α and β to be roots of the equation $x^2-Bx-A=0$;

then $$\sqrt{x^2-Bx-A}=\sqrt{(x-\alpha)(x-\beta)},$$

and $$\sqrt{A+Bx-x^2}=\sqrt{(x-\alpha)(\beta-x)}.$$

Let $$\sqrt{(x-\alpha)(\beta-x)}=(x-\alpha)\,z;$$

then $$(x-\alpha)(\beta-x)=(x-\alpha)^2z^2,$$

$$x=\frac{\alpha z^2+\beta}{z^2+1},$$

and $$dx=\frac{2z(\alpha-\beta)\,dz}{(z^2+1)^2}.$$

Therefore $$\sqrt{A+Bx-x^2}=(x-\alpha)\,z=\frac{(\beta-\alpha)\,z}{z^2+1}.$$

Thus the given surd is expressed in rational terms of another variable.

PROBLEMS.

1. Find $\int \frac{dx}{\sqrt{A + Bx + x^2}}$.

Substituting the values of dx and $\sqrt{A + Bx + x^2}$ as found in Case 1, gives

$$\int \frac{dx}{\sqrt{A + Bx + x^2}} = \int \frac{2(z^2 + Bz + A)\,dz \times (2z + B)}{(B + 2z)^2 \times (z^2 + Bz + A)}$$

$$= \int \frac{2\,dz}{B + 2z} = \log\left(\frac{B}{2} + z\right)$$

$$= \log\left[\frac{B}{2} + x + \sqrt{A + Bx + x^2}\right]. \qquad (1)$$

If $B = 0$, (1) becomes

$$\int \frac{dx}{\sqrt{A + x^2}} = \log\left[x + \sqrt{A + x^2}\right]. \qquad (2)$$

If $A = 1$, (2) becomes

$$\int \frac{dx}{\sqrt{1 + x^2}} = \log\left[x + \sqrt{1 + x^2}\right]. \qquad (3)$$

2. Find $\int \frac{dx}{\sqrt{A + Bx - x^2}}$.

Substituting the values of dx and $\sqrt{A + Bx - x^2}$ as found in Case 2, gives

$$\int \frac{dx}{\sqrt{A + Bx - x^2}} = \int \frac{2(\alpha - \beta) z\,dz\,(z^2 + 1)}{(z^2 + 1)^2 (\beta - \alpha) z} = -\int \frac{2\,dz}{1 + z^2}$$

$$= -2 \operatorname{arc} \tan z = -2 \operatorname{arc} \tan \sqrt{\frac{\beta - x}{x - \alpha}}.$$

3. $\int \frac{dx}{\sqrt{2 + 3x + x^2}} = \log\left[3 + 2x + 2\sqrt{2 + 3x + x^2}\right]$.

4. $\int \frac{dx}{\sqrt{5x - 6 - x^2}} = 2 \operatorname{arc} \cot \left(\frac{3 - x}{x - 2}\right)^{\frac{1}{2}}$.

5. $\int \frac{dx}{\sqrt{x^2 + x}} = \log\left(\frac{1}{2} + x + \sqrt{x^2 + x}\right)$.

6. $\displaystyle\int\frac{dx}{(1+x^2)\sqrt{1-x^2}}=\frac{1}{\sqrt{2}}\operatorname{arc}\tan\left(\frac{x\sqrt{2}}{\sqrt{1-x^2}}\right).$

7. $\displaystyle\int\frac{dx}{\sqrt{a^2+b^2x^2}}=\frac{1}{b}\log\left(bx+\sqrt{a^2+b^2x^2}\right).$

8. $\displaystyle\int\frac{dx}{x\sqrt{a^2+b^2x^2}}=\frac{1}{a}\log\left(\frac{\sqrt{a^2+b^2x^2}-a}{bx}\right),$

$$\text{or } -\frac{1}{a}\log\left(\frac{\sqrt{a^2+b^2x^2}+a}{x}\right).$$

Art. 106. Binomial Differentials.

Binomial differentials have the form $x^m(a+bx^n)^p dx$, in which m, n and p are any numbers, positive or negative, integral or fractional.

1st. If m and n are fractional, and the differential has the form

$$x^{\frac{s}{t}}(a+bx^{\frac{l}{r}})^p dx,$$

$x=z^{rt}$ may be substituted, and the expression becomes

$$z^{rs}(a+bz^{lt})^p rt\,z^{rt-1}dz=rt\,z^{rs+rt-1}(a+bz^{lt})^p dz.$$

2d. If n is negative, and the differential has the form

$$x^m(a+bx^{-n})^p dx,$$

$x=\dfrac{1}{z}$ may be substituted, and the expression becomes

$$x^m(a+bx^{-n})^p dx=-z^{-m-2}(a+bz^n)^p dz.$$

Hence, any binomial differential may be transformed into another, having integers for the exponents of the variable, and having a positive exponent for the variable within the parenthesis.

In the following articles every binomial differential is assumed to have this reduced form.

Art. 107. Conditions of Integrability of Binomial Differentials.

As the exponent of the parenthesis is any number, let it be represented by $\dfrac{p}{q}$, and the form may be written

$$x^m(a+bx^n)^{\frac{p}{q}}dx. \qquad (1)$$

1st. Assume $$(a + bx^n) = z^q;$$

then $$(a + bx^n)^{\frac{p}{q}} = z^p, \tag{2}$$

$$x = \left(\frac{z^q - a}{b}\right)^{\frac{1}{n}},$$

$$x^m = \left(\frac{z^q - a}{b}\right)^{\frac{m}{n}}, \tag{3}$$

and $$dx = \frac{q}{nb} z^{q-1}\left(\frac{z^q - a}{b}\right)^{\frac{1}{n}-1} dz. \tag{4}$$

Substituting the values from (2), (3) and (4) in (1),

$$x^m (a + bx^n)^{\frac{p}{q}} dx = \frac{q}{nb} z^{p+q-1}\left(\frac{z^q - a}{b}\right)^{\frac{m+1}{n}-1} dz, \tag{5}$$

which is rational when $\frac{m+1}{n}$ is an integer or 0.

2d. Assume $$ax^{-n} + b = z^q; \tag{1}$$

then $$x = a^{\frac{1}{n}} (z^q - b)^{-\frac{1}{n}},$$

$$x^n = a (z^q - b)^{-1}, \tag{2}$$

$$x^m = a^{\frac{m}{n}} (z^q - b)^{-\frac{m}{n}}, \tag{3}$$

and $$dx = -\frac{q}{n} a^{\frac{1}{n}} (z^q - b)^{-\frac{1}{n}-1} z^{q-1} dz. \tag{4}$$

Multiplying (2) by b, adding a, and taking the $\frac{p}{q}$ power,

$$(a + bx^n)^{\frac{p}{q}} = a^{\frac{p}{q}} (z^q - b)^{-\frac{p}{q}} z^p. \tag{5}$$

Taking the product of (3), (4) and (5), gives

$$x^m (a + bx^n)^{\frac{p}{q}} dx = -\frac{q}{n} a^{\frac{m}{n}+\frac{p}{q}+\frac{1}{n}} (z^q - b)^{-\left(\frac{m+1}{n}+\frac{p}{q}+1\right)} z^{p+q-1} dz,$$

which is rational when $\frac{m+1}{n} + \frac{p}{q}$ is an integer or 0. Therefore, the binomial differential can be rationalized:

1st. *When the exponent of the variable without the parenthesis, increased by one, is exactly divisible by the exponent of the variable within the parenthesis.*

2d. *When this fraction increased by the exponent of the parenthesis is an integer.*

PROBLEMS.

1. Find $\int x^5 (2 + 3x^2)^{\frac{1}{2}} dx.$

Here $$\frac{m+1}{n} - 1 = \frac{5+1}{2} - 1 = 2;$$

therefore the first condition of integrability is satisfied.

Hence, let $$(2 + 3x^2) = z^2;$$

then $$(2 + 3x^2)^{\frac{1}{2}} = z,$$

$$x^5 = \left(\frac{z^2 - 2}{3}\right)^{\frac{5}{2}},$$

and $$dx = \frac{z\,dz}{3\left(\frac{z^2-2}{3}\right)^{\frac{1}{2}}}.$$

Therefore $$\int x^5 (2 + 3x^2)^{\frac{1}{2}} dx = \int \left(\frac{z^2 - 2}{3}\right)^{\frac{5}{2}} \cdot z \cdot \frac{z\,dz}{3\left(\frac{z^2-2}{3}\right)^{\frac{1}{2}}}$$

$$= \frac{1}{27}\int (z^6 - 4z^4 + 4z^2)\,dz$$

$$= \frac{1}{27}\left(\frac{z^7}{7} - \frac{4z^5}{5} + \frac{4z^3}{3}\right)$$

$$= \tfrac{1}{27}\left[\tfrac{1}{7}(2 + 3x^2)^{\frac{7}{2}} - \tfrac{4}{5}(2 + 3x^2)^{\frac{5}{2}} + \tfrac{4}{3}(2 + 3x^2)^{\frac{3}{2}}\right].$$

2. $\int x^{-4} (1 + x^2)^{-\frac{1}{2}} dx = \frac{(1 + x^2)^{\frac{1}{2}}(2x^2 - 1)}{3x^3}.$

3. $\int x^3 (a + bx^2)^{\frac{3}{2}} dx = (a + bx^2)^{\frac{5}{2}}\left(\frac{5bx^2 - 2a}{35b^2}\right).$

4. $\int \frac{dx}{x^2 (1 + x^2)^{\frac{3}{2}}} = -\left(2x + \frac{1}{x}\right)(1 + x^2)^{-\frac{1}{2}}.$

5. $\int x^{-2}(a+x^3)^{-\frac{5}{3}}dx = -\frac{3x^3+2a}{2a^2x(a+x^3)^{\frac{2}{3}}}.$

6. $\int (1+x^2)^{\frac{1}{2}}x^3dx = \frac{(3x^2-2)(1+x^2)^{\frac{3}{2}}}{15}.$

7. $\int x^5(a+x^2)^{\frac{1}{3}}dx = \frac{3}{20}(a+x^2)^{\frac{10}{3}} - \frac{3}{7}(a+x^2)^{\frac{7}{3}}a + \frac{3}{8}(a+x^2)^{\frac{4}{3}}a^2.$

PRACTICAL PROBLEM.

A vessel in the form of a right circular cone is filled with water and placed with its axis vertical and vertex down. If the height $= h$, and radius of the base $= r$, how long will it require to empty itself through an orifice in the vertex of the area a?

Neglecting the resistances, if the vessel is kept always full, the velocity of discharge through an orifice in the bottom is that due to a body falling from a height equal to the depth of the water. If v denotes the velocity and x the depth of the water,

$$v = \sqrt{2gx}.$$

If dQ denotes the quantity discharged in the time dt through an orifice of the area a,

$$dQ = adt\sqrt{2gx}.$$

But in the time dt the surface whose area is S has descended the distance dx, thus

$$dQ = Sdx.$$

Hence $$Sdx = adt\sqrt{2gx},$$

or $$dt = \frac{Sdx}{a\sqrt{2gx}}.$$

At the distance x from the vertex,

$$S = \frac{\pi r^2x^2}{h^2}.$$

Therefore $$t = \int_0^h \frac{\pi r^2x^2dx}{ah^2\sqrt{2gx}} = \frac{2\pi r^2\sqrt{h}}{5a\sqrt{2g}}.$$

CHAPTER XVII.

INTEGRATION BY PARTS, AND BY SUCCESSIVE REDUCTION.

ART. 108. INTEGRATION BY PARTS.

Integrating both members of $d(uv) = u\,dv + v\,du$, and transposing,

$$\int u\,dv = uv - \int v\,du. \tag{1}$$

Equation (1) is the *formula for integration by parts.*

By this formula, $\int u\,dv$ is made to depend on $\int v\,du$, and this new integral is frequently much simpler than the given one.

PROBLEMS.

1. Find $\int x^n \log x\,dx$.

Let $$u = \log x, \text{ and } dv = x^n dx;$$

then $$du = \frac{dx}{x}, \quad \text{and} \quad v = \frac{x^{n+1}}{n+1}.$$

Substituting these values in the formula for integration by parts,

$$\int x^n \log x\,dx = \frac{x^{n+1}\log x}{n+1} - \frac{1}{n+1}\int x^n dx$$

$$= \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right).$$

2. $\int \log x\,dx = x(\log x - 1)$.

3. $\int \theta \sin\theta\,d\theta = -\theta\cos\theta + \sin\theta$.

4. $\int \frac{\log(\log x)\,dx}{x} = \log x \cdot \log(\log x) - \log x.$

5. $\int xe^{ax}dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right).$

6. $\int \arcsin x\,dx = x \arcsin x + (1 - x^2)^{\frac{1}{2}}.$

7. $\int x \cos x\,dx = x \sin x + \cos x.$

8. $\int x \tan^2 x\,dx = x \tan x - \frac{x^2}{2} + \log \cos x.$

9. $\int \frac{\log x\,dx}{(x+1)^2} = \frac{x}{x+1}\log x - \log(x+1).$

10. $\int x^3 e^{ax}dx = \left(x^3 - \frac{3x^2}{a} + \frac{6x}{a^2} - \frac{6}{a^3}\right)\frac{e^{ax}}{a}.$

Art. 109. Formulas of Reduction.

When the integral, $\int x^m(a + bx^n)^p dx,$

satisfies either of the conditions of integrability as given in Art. 107, it may be rationalized as explained in that article and then integrated.

But by means of certain formulas of reduction, derived by the aid of the formula for integration by parts, the given expression may be made to depend upon simpler integrals of the same form. This method is called *integration by successive reduction*, and the integrals given by this method are generally in convenient form for integration between limits.

1. *Formula A.*

Assume $\int x^m(a + bx^n)^p dx = \int u\,dv = uv - \int v\,du.$ (1)

Let $dv = x^{n-1}(a + bx^n)^p dx,$ then $u = x^{m-n+1}.$

Hence $v = \frac{(a + bx^n)^{p+1}}{nb(p+1)},$ and $du = (m - n + 1)x^{m-n}dx.$

Substituting in (1), and putting $a + bx^n = X$,

$$\int x^m X^p dx = \frac{x^{m-n+1}X^{p+1}}{nb(p+1)} - \frac{m-n+1}{nb(p+1)}\int x^{m-n}X^{p+1}dx. \tag{2}$$

Now $\int x^{m-n}X^{p+1}dx = \int x^{m-n}X^p(a + bx^n)\,dx$

$$= a\int x^{m-n}X^p dx + b\int x^m X^p dx. \tag{3}$$

Substituting in the right member of (2) the value from (3),

$$\int x^m X^p dx = \frac{x^{m-n+1}X^{p+1}}{nb(p+1)} - \frac{(m-n+1)a}{nb(p+1)}\int x^{m-n}X^p dx$$

$$- \frac{m-n+1}{n(p+1)}\int x^m X^p dx.$$

Transposing the last term to the first member and solving for $\int x^m X^p dx$,

$$\int x^m X^p dx = \frac{x^{m-n+1}X^{p+1}}{b(np+m+1)} - \frac{a(m-n+1)}{b(np+m+1)}\int x^{m-n}X^p dx. \tag{A}$$

By Formula (A), the given integral is made to depend upon another of a similar form, having the exponent of x without the parenthesis diminished by the exponent of x within.

2. *Formula B.*

$$\int x^m X^p dx = \int x^m X^{p-1}(a + bx^n)\,dx$$

$$= a\int x^m X^{p-1}dx + b\int x^{m+n}X^{p-1}dx. \tag{1}$$

Applying Formula (A) to the last term of (1), by substituting $m + n$ for m, and $p - 1$ for p,

$$b\int x^{m+n}X^{p-1}dx = \frac{x^{m+1}X^p}{np+m+1} - \frac{a(m+1)}{np+m+1}\int x^m X^{p-1}dx,$$

which substituted in (1), by uniting similar terms, gives

$$\int x^m X^p dx = \frac{x^{m+1}X^p}{np+m+1} + \frac{anp}{np+m+1}\int x^m X^{p-1}dx. \tag{B}$$

By Formula (B), the given integral is made to depend upon another of a similar form, having the exponent of the parenthesis diminished by unity.

Formulas (A) and (B) fail when $np + m + 1 = 0$, but in this case $\frac{m+1}{n} + p = 0$; hence the method of integration of Art. 107 is applicable.

3. *Formula C.*

Solving Formula (A) for $\int x^{m-n}X^p dx$, gives

$$\int x^{m-n}X^p dx = \frac{x^{m-n+1}X^{p+1}}{a(m-n+1)} - \frac{b(np+m+1)}{a(m-n+1)}\int x^m X^p dx.$$

Substituting $-m$ for $m-n$,

$$\int x^{-m}X^p dx = \frac{x^{-m+1}X^{p+1}}{-a(m-1)} + \frac{b(m-n-np-1)}{-a(m-1)}\int x^{-m+n}X^p dx. \qquad (C)$$

By Formula (C), the given integral is made to depend upon another of a similar form, having the exponent of x without the parenthesis increased by the exponent of x within.

Formula (C) fails when $m - 1 = 0$; in this case

$$m = 1, \text{ and } -m+1 = 0;$$

hence the method of integration of Art. 107 is applicable.

4. *Formula D.*

Solving Formula (B) for $\int x^m X^{p-1} dx$, gives

$$\int x^m X^{p-1} dx = -\frac{x^{m+1}X^p}{anp} + \frac{(np+m+1)}{anp}\int x^m X^p dx.$$

Substituting $-p$ for $p-1$,

$$\int x^m X^{-p} dx = \frac{x^{m+1}X^{-p+1}}{an(p-1)} - \frac{(m+n-np+1)}{an(p-1)}\int x^m X^{-p+1} dx. \qquad (D)$$

By Formula (D), the given integral is made to depend upon another of a similar form, having the exponent of the parenthesis increased by unity.

Formula (D) fails when $p - 1 = 0$, but in this case the integral reduces to a fundamental form.

PROBLEMS.

1. Find $\int \frac{x^3 dx}{\sqrt{a^2 - x^2}}$.

$$\int \frac{x^3 dx}{\sqrt{a^2 - x^2}} = \int x^3 (a^2 - x^2)^{-\frac{1}{2}} dx.$$

Here $m = 3$, $n = 2$, $p = -\frac{1}{2}$, $a = a^2$, and $b = -1$.

Substituting these values in (A),

$$\int x^3 (a^2 - x^2)^{-\frac{1}{2}} dx = \frac{x^2 (a^2 - x^2)^{\frac{1}{2}}}{-3} - \frac{2a^2}{3}\int x (a^2 - x^2)^{-\frac{1}{2}} dx$$

$$= -\frac{x^2 (a^2 - x^2)^{\frac{1}{2}}}{3} + \frac{2a^2}{3}[-(a^2 - x^2)^{\frac{1}{2}}]$$

$$= -\tfrac{1}{3}(a^2 - x^2)^{\frac{1}{2}}(x^2 + 2a^2).$$

2. Find $\int \frac{dx}{(1 + x^2)^3}$.

$$\int \frac{dx}{(1 + x^2)^3} = \int (1 + x^2)^{-3} dx.$$

Here $m = 0$, $n = 2$, $-p = -3$, $a = 1$, and $b = 1$.

Substituting these values in (D),

$$\int (1 + x^2)^{-3} dx = \frac{x(1 + x^2)^{-2}}{4} + \tfrac{3}{4}\int (1 + x^2)^{-2} dx. \tag{1}$$

Applying (D) to the last term of (1), making $m=0$, $n=2$, $-p = -2$, $a = 1$, and $b = 1$,

$$\int (1 + x^2)^{-2} dx = \frac{x(1 + x^2)^{-1}}{2} + \tfrac{1}{2}\int (1 + x^2)^{-1} dx$$

$$= \frac{x}{2(1 + x^2)} + \tfrac{1}{2} \operatorname{arc} \tan x.$$

Therefore $\int \frac{dx}{(1 + x^2)^3} = \frac{x}{4(1 + x^2)^2} + \frac{3}{8}\frac{x}{(1 + x^2)} + \tfrac{3}{8} \operatorname{arc} \tan x.$

3. Find $\int \frac{dx}{x^3 \sqrt{a^2 - x^2}}$.

$$\int \frac{dx}{x^3 \sqrt{a^2 - x^2}} = \int x^{-3} (a^2 - x^2)^{-\frac{1}{2}} dx.$$

Here $-m=-3$, $n=2$, $p=-\frac{1}{2}$, $a=a^2$, and $b=-1$.

Substituting these values in (C),

$$\int x^{-3}(a^2-x^2)^{-\frac{1}{2}}dx=-\frac{x^{-2}(a^2-x^2)^{\frac{1}{2}}}{2a^2}+\frac{1}{2a^2}\int x^{-1}(a^2-x^2)^{-\frac{1}{2}}dx.$$

By Art. 105, Ex. 8,

$$\int x^{-1}(a^2-x^2)^{-\frac{1}{2}}dx=\frac{1}{a}\log\left(\frac{a-\sqrt{a^2-x^2}}{x}\right).$$

Therefore $$\int\frac{dx}{x^3\sqrt{a^2-x^2}}=-\frac{\sqrt{a^2-x^2}}{2a^2x^2}+\frac{1}{2a^3}\log\left(\frac{a-\sqrt{a^2-x^2}}{x}\right).$$

4. $\int(a^2-x^2)^{\frac{1}{2}}dx=\frac{x(a^2-x^2)^{\frac{1}{2}}}{2}+\frac{a^2}{2}\operatorname{arc}\sin\frac{x}{a}.$

5. $\int\frac{x^2dx}{\sqrt{a^2-x^2}}=-\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\operatorname{arc}\sin\frac{x}{a}.$

6. $\int(1-x^2)^{\frac{3}{2}}dx=\frac{1}{4}x(1-x^2)^{\frac{3}{2}}+\frac{3}{8}x(1-x^2)^{\frac{1}{2}}+\frac{3}{8}\operatorname{arc}\sin x.$

7. $\int\sqrt{(a^2+x^2)}dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log(x+\sqrt{a^2+x^2}).$

8. $\int\frac{x^5dx}{\sqrt{1-x^2}}=-\left(\frac{x^4}{5}+\frac{4x^2}{5\cdot3}+\frac{4\cdot2}{5\cdot3}\right)\sqrt{1-x^2}.$

9. $\int x^2(1-x^2)^{\frac{1}{2}}dx=\frac{1}{8}x(2x^2-1)(1-x^2)^{\frac{1}{2}}+\frac{1}{8}\operatorname{arc}\sin x.$

10. $\int x^3(1+x^2)^{\frac{3}{2}}dx=\frac{(5x^2-2)}{35}(1+x^2)^{\frac{5}{2}}.$

11. $\int\frac{dx}{x^2\sqrt{1+x}}=-\frac{\sqrt{1+x}}{x}-\frac{1}{2}\log\left(\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right).$

12. $\int\frac{dx}{(a+bx^2)^{\frac{5}{2}}}=\left(\frac{1}{a+bx^2}+\frac{2}{a}\right)\frac{x}{3a\sqrt{a+bx^2}}.$

13. $\int\frac{x^3dx}{\sqrt{2ax-x^2}}=-\left(\frac{x^2}{3}+\frac{5ax}{6}+\frac{5a^2}{2}\right)\sqrt{2ax-x^2}+\frac{5}{2}a^3\operatorname{arc}\operatorname{vers}\frac{x}{a}.$

Remark. Reduce $\int\frac{x^3dx}{\sqrt{2ax-x^2}}$ to the form $\int x^{\frac{5}{2}}(2a-x)^{-\frac{1}{2}}dx$

14. $\int \frac{x^2 dx}{\sqrt{2ax - x^2}} = -\frac{x + 3a}{2}\sqrt{2ax - x^2} + \frac{3}{2}a^2 \text{ arc vers } \frac{x}{a}.$

15. $\int \frac{x^4 dx}{(1 - x^2)^{\frac{3}{2}}} = \frac{x(3 - x^2)}{2(1 - x^2)^{\frac{1}{2}}} - \frac{3}{2}\text{ arc sin } x.$

16. $\int \frac{x^8 dx}{\sqrt{1 - x^3}} = -\frac{2}{45}(3x^6 + 4x^3 + 8)\sqrt{1 - x^3}.$

CHAPTER XVIII.

INTEGRATION OF TRANSCENDENTAL FUNCTIONS. INTEGRATION BY SERIES.

ART. 110. INTRODUCTION.

The method of integration by parts gives important reduction formulas for transcendental functions. Only a comparatively small number of logarithmic and exponential functions can be integrated by general methods. It is frequently necessary to resort to methods of approximation. Some of the principal integrable forms will be given in this chapter.

ART. 111. INTEGRATION OF THE FORM $\int f(x)(\log x)^n dx$.

It is assumed in this form that $f(x)$ is an algebraic function and n is a positive integer.

Let $$f(x)dx = dv, \text{ and } (\log x)^n = u;$$

then $$\int f(x)dx = v, \text{ and } n(\log x)^{n-1}\frac{dx}{x} = du.$$

Substituting these values in $\int u\,dv = uv - \int v\,du$,

$$\int f(x)(\log x)^n dx = (\log x)^n \int f(x)\,dx - \int \left[n(\log x)^{n-1}\frac{dx}{x}\int f(x)\,dx\right];$$

or, by making $$\int f(x)\,dx = X,$$

$$\int f(x)(\log x)^n dx = X(\log x)^n - n\int \frac{X}{x}(\log x)^{n-1}dx. \qquad (1)$$

Hence, whenever it is possible to integrate the factor $f(x)dx$, the given integral will depend upon another of a similar form, in which the exponent of the logarithm is diminished by unity. By repeated

applications of this formula the given integral will depend finally on the algebraic form $\int \phi(x)\,dx$.

PROBLEMS.

1. Find $\int x(\log x)^2 dx$.

Let $$x\,dx = dv, \text{ and } (\log x)^2 = u;$$

then $$\frac{x^2}{2} = v, \text{ and } 2\log x \frac{dx}{x} = du.$$

Hence $$\int x(\log x)^2 dx = \frac{x^2}{2}(\log x)^2 - \int x^2 \log x \frac{dx}{x}.$$

Similarly, $$\int x \log x\,dx = \frac{x^2}{2}(\log x) - \int \frac{x^2}{2}\frac{dx}{x}.$$

Therefore $$\int x(\log x)^2 dx = \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{x^2}{4}.$$

2. $$\int x^4 (\log x)^2 dx = \frac{x^5}{5}(\log^2 x - \tfrac{2}{5}\log x + \tfrac{2}{25}).$$

3. $$\int \frac{\log x\,dx}{(1-x)^2} = \frac{x\log x}{1-x} + \log(1-x).$$

4. $$\int \frac{(\log x)^2 dx}{x^{\frac{5}{2}}} = -\frac{2}{3\sqrt{x^3}}[(\log x)^2 + \tfrac{4}{3}\log x + \tfrac{8}{9}].$$

5. $$\int x^n (\log x)^2 dx = \frac{x^{n+1}}{n+1}\left[(\log x)^2 - \frac{2}{n+1}\log x + \frac{2}{(n+1)^2}\right].$$

Art. 112. Integration of the Form $\int x^m a^{nx} dx$.

In this form it is assumed that m is a positive integer.

Let $$a^{nx}dx = dv, \text{ and } x^m = u;$$

then $$\frac{a^{nx}}{n\log a} = v, \text{ and } mx^{m-1}dx = du.$$

Therefore $$\int x^m a^{nx} dx = \frac{x^m a^{nx}}{n\log a} - \frac{m}{n\log a}\int x^{m-1}a^{nx}dx.$$

By successive applications of this formula, the exponent of x can be finally reduced to zero, and the given integral made to depend on the known form, $\int a^{nx}dx$.

PROBLEMS.

1. Find $\int x^2 e^{ax} dx$.

Let $$e^{ax}dx = dv, \text{ and } x^2 = u;$$

then $$\int x^2 e^{ax} dx = \frac{1}{a} e^{ax} x^2 - \frac{2}{a} \int x e^{ax} dx.$$

Similarly, $$\int x e^{ax} dx = \frac{1}{a} e^{ax} x - \frac{1}{a} \int e^{ax} dx.$$

Therefore $$\int x^2 e^{ax} dx = \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right).$$

2. $\int x a^x dx = \frac{a^x}{\log a}\left(x - \frac{1}{\log a}\right).$

3. $\int x^2 e^x dx = e^x (x^2 - 2x + 2).$

4. $\int x^3 e^{ax} dx = \frac{e^{ax}}{a}\left(x^3 - \frac{3}{a} x^2 + \frac{6}{a^2} x - \frac{6}{a^3}\right).$

5. $\int \frac{x^2 dx}{e^x} = -e^{-x} (x^2 + 2x + 2).$

Art. 113. Integration of the Form $\int \sin^m \theta \cos^n \theta \, d\theta$.

1st. When either m or n, or both, are odd positive integers.

In this case the integration can be effected as in the following example:

$$\int \sin^3 \theta \cos^4 \theta \, d\theta = \int (1 - \cos^2 \theta) \cos^4 \theta \sin \theta \, d\theta$$
$$= -\int (\cos^4 \theta - \cos^6 \theta) \, d \cos \theta$$
$$= -\frac{\cos^5 \theta}{5} + \frac{\cos^7 \theta}{7}.$$

2d. When $m + n$ is an even negative integer.

In this case the integration can be effected as in the following example:

$$\begin{aligned}\int \sin^2\theta\cos^{-6}\theta\,d\theta &= \int \tan^2\theta\cos^{-4}\theta\,d\theta\\ &= \int \tan^2\theta\sec^4\theta\,d\theta\\ &= \int \tan^2\theta\,(1+\tan^2\theta)\cdot d\tan\theta\\ &= \int (\tan^2\theta\cdot d\tan\theta + \tan^4\theta\cdot d\tan\theta)\\ &= \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5}.\end{aligned}$$

3d. When the form is not immediately integrable, neither of the aforesaid conditions being fulfilled.

In this case the integral can be obtained by successive reductions.

Let $\quad \sin\theta = x;$ then $\sin^m\theta = x^m,\ \cos^n\theta = (1-x^2)^{\frac{n}{2}},$

and $$d\theta = (1-x^2)^{-\frac{1}{2}}dx.$$

Hence $$\int \sin^m\theta\cos^n\theta\,d\theta = \int x^m(1-x^2)^{\frac{n-1}{2}}dx. \tag{1}$$

Thus the given trigonometric form may be transformed into a binomial differential which may be integrated by means of the formulas of reduction.

For example, to find $\int \sin^4\theta\cos^4\theta\,d\theta.$

Let $\sin\theta = x;$ then $\sin^4\theta = x^4,\ \cos^4\theta = (1-x^2)^2,$ and

$$d\theta = (1-x^2)^{-\frac{1}{2}}dx.$$

Hence $$\int \sin^4\theta\cos^4\theta\,d\theta = \int x^4(1-x^2)^{\frac{3}{2}}dx.$$

Applying Formula (A) twice,

$$\int x^4(1-x^2)^{\frac{3}{2}}dx = -\frac{x^3(1-x^2)^{\frac{5}{2}}}{8} - \tfrac{3}{8}\cdot\tfrac{1}{6}\,x(1-x^2)^{\frac{5}{2}} + \tfrac{3}{8}\cdot\tfrac{1}{6}\int(1-x^2)^{\frac{3}{2}}dx. \tag{2}$$

Applying Formula (B) twice to the last term of (2),

$$\int(1-x^2)^{\frac{3}{2}}dx = \frac{x(1-x^2)^{\frac{3}{2}}}{4} + \tfrac{3}{4}\cdot\tfrac{1}{2}x(1-x^2)^{\frac{1}{2}} + \tfrac{3}{4}\cdot\tfrac{1}{2}\int(1-x^2)^{-\frac{1}{2}}dx.$$

Hence

$$\int x^4(1-x^2)^{\frac{3}{2}}dx = -\frac{x^3(1-x^2)^{\frac{5}{2}}}{8} - \tfrac{3}{8}\cdot\tfrac{1}{6}x(1-x^2)^{\frac{5}{2}} + \tfrac{3}{8}\cdot\tfrac{1}{6}\cdot\tfrac{1}{4}x(1-x^2)^{\frac{3}{2}}$$
$$+ \tfrac{3}{8}\cdot\tfrac{1}{6}\cdot\tfrac{3}{4}\cdot\tfrac{1}{2}x(1-x^2)^{\frac{1}{2}} + \tfrac{3}{8}\cdot\tfrac{1}{6}\cdot\tfrac{3}{4}\cdot\tfrac{1}{2}\operatorname{arc}\sin x.$$

Therefore

$$\int \sin^4\theta\cos^4\theta\,d\theta = -\frac{\cos^5\theta}{8}(\sin^3\theta + \tfrac{1}{6}\sin\theta) + \tfrac{1}{64}\sin\theta(\cos^3\theta + \tfrac{3}{2}\cos\theta)$$
$$+ \tfrac{3}{128}\theta.$$

PROBLEMS.

1. $\int \sin^5\theta\cos^5\theta\,d\theta = -\left(\frac{\cos^6\theta}{6} - \frac{\cos^8\theta}{4} + \frac{\cos^{10}\theta}{10}\right).$

2. $\int \frac{d\theta}{\cos^6\theta} = \tan\theta + \tfrac{2}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta.$

3. $\int \frac{\sin^5\theta\,d\theta}{\cos^2\theta} = \sec\theta + 2\cos\theta - \tfrac{1}{3}\cos^3\theta.$

4. $\int \frac{d\theta}{\sin^4\theta\cos^2\theta} = \frac{1}{\cos\theta\sin^3\theta} - \frac{4\cos\theta}{3\sin^3\theta} - \frac{8\cos\theta}{3\sin\theta}.$

5. $\int \sin^3\theta\cos^3\theta\,d\theta = \tfrac{1}{4}\sin^4\theta - \tfrac{1}{6}\sin^6\theta.$

6. $\int \frac{\sin^2\theta}{\cos^6\theta}d\theta = \frac{\tan^5\theta}{5} + \frac{\tan^3\theta}{3}.$

7. $\int \sin^4\theta\,d\theta = -\tfrac{1}{4}\cos\theta(\sin^3\theta + \tfrac{3}{2}\sin\theta) + \tfrac{3}{8}\theta.$

8. $\int \cos^4\theta\,d\theta = \tfrac{1}{4}\sin\theta(\cos^3\theta + \tfrac{3}{2}\cos\theta) + \tfrac{3}{8}\theta.$

9. $\int \cos^2\theta\sin^4\theta\,d\theta = \frac{\sin\theta\cos\theta}{2}\left(\frac{\sin^4\theta}{3} - \frac{\sin^2\theta}{12} - \frac{1}{8}\right) + \frac{\theta}{16}.$

10. $\int \frac{d\theta}{\sin\theta\cos^2\theta} = \sec\theta + \log\tan\frac{\theta}{2}.$

11. $\int \sec^3\theta\,d\theta = \frac{\sec\theta\tan\theta}{2} + \tfrac{1}{2}\log(\sec\theta + \tan\theta).$

12. $\int \frac{d\theta}{\sin^3\theta} = -\frac{\cos\theta}{2\sin^2\theta} + \tfrac{1}{2}\log\tan\frac{\theta}{2}.$

ART. 114. INTEGRATION OF THE FORMS $\int x^n \cos(ax)\,dx$ AND $\int x^n \sin(ax)\,dx$.

The formula for integration by parts is used, assuming that $u = x^n$. Evidently, each application of the formula will diminish the exponent n by one; therefore, when n is a positive integer, the given form may be made to depend finally on $\int \sin(ax)\,dx$ or $\int \cos(ax)\,dx$, each being a simple known form.

For example, to find $\int x^2 \sin x\,dx$.

Assume $\qquad u = x^2$, and $dv = \sin x\,dx$;

then $\qquad du = 2x\,dx$, and $v = -\cos x$.

Hence $\qquad \int x^2 \sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx.$

Similarly,
$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$
$$= x\sin x + \cos x.$$

Therefore $\int x^2 \sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x.$

ART. 115. INTEGRATION OF THE FORMS $\int e^{ax}\sin^n x\,dx$ AND $\int e^{ax}\cos^n x\,dx$.

Assume $\qquad u = \sin^n x$, and $dv = e^{ax}\,dx$;

then $\qquad du = n\sin^{n-1} x\cos x\,dx$, and $v = \dfrac{e^{ax}}{a}$.

Hence
$$\int e^{ax}\sin^n x\,dx = \frac{1}{a}e^{ax}\sin^n x - \frac{n}{a}\int e^{ax}\sin^{n-1} x\cos x\,dx. \qquad (1)$$

Again, assume $\qquad u = \sin^{n-1} x\cos x$, and $dv = e^{ax}\,dx$;

then
$$du = (n-1)\sin^{n-2} x\cos^2 x\,dx - \sin^n x\,dx$$
$$= (n-1)\sin^{n-2} x\,dx - n\sin^n x\,dx,$$

and $\qquad v = \dfrac{e^{ax}}{a}$.

Hence

$$\int e^{ax}\sin^{n-1}x\cos x\,dx = \frac{1}{a}e^{ax}\sin^{n-1}x\cos x - \frac{n-1}{a}\int e^{ax}\sin^{n-2}x\,dx$$
$$+\frac{n}{a}\int e^{ax}\sin^n x\,dx.$$

Substituting in (1), and solving for $\int e^{ax}\sin^n x\,dx$, gives

$$\int e^{ax}\sin^n x\,dx = \frac{e^{ax}\sin^{n-1}x\,(a\sin x - n\cos x)}{n^2+a^2}$$
$$+\frac{n(n-1)}{n^2+a^2}\int e^{ax}\sin^{n-2}x\,dx. \qquad (2)$$

Each application of this formula diminishes the exponent of $\sin x$ by 2. By repeated applications n can be reduced to 0 or 1, and the given integral will finally depend on

$$\int e^{ax}\,dx = \frac{e^{ax}}{a}, \text{ or } \int e^{ax}\sin x\,dx.$$

The value of $\int e^{ax}\sin x\,dx$ is obtained directly from (2) by making $n = 1$.

In like manner $\int e^{ax}\cos^n x\,dx$ can be obtained.

PROBLEMS.

1. $\int x^2\cos x\,dx = x^2\sin x + 2x\cos x - 2\sin x.$

2. $\int x^3\sin x\,dx = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x.$

3. $\int e^{ax}\sin x\,dx = \dfrac{e^{ax}}{a^2+1}(a\sin x - \cos x).$

4. $\int e^{x}\sin^3 x\,dx = \dfrac{e^x}{10}(\sin^3 x + 3\cos^3 x + 3\sin x - 6\cos x).$

5. $\int e^{ax}\cos^2 x\,dx = \dfrac{e^{ax}\cos x\,(a\cos x + 2\sin x)}{4+a^2} + \dfrac{2}{4+a^2}\cdot\dfrac{e^{ax}}{a}.$

ART. 116. INTEGRATION OF THE FORMS

$$\int f(x) \text{ arc } \sin x dx, \quad \int f(x) \text{ arc } \cos x dx, \quad \int f(x) \text{ arc } \tan x dx, \text{ etc.}$$

In these forms, $f(x)$ is an algebraic function.

Any one of these forms may be integrated by using the formula for integration by parts, assuming $dv = f(x)dx$. For example, to find

$$\int x^2 \text{ arc } \sin x dx.$$

Assume $\qquad dv = x^2dx$, and $u = \text{arc } \sin x$;

then $\qquad v = \frac{x^3}{3}$, and $du = \frac{dx}{\sqrt{1-x^2}}$.

Hence $\qquad \int x^2 \text{ arc } \sin x dx = \frac{x^3}{3} \text{ arc } \sin x - \frac{1}{3}\int \frac{x^3dx}{\sqrt{1-x^2}}$.

Substituting $a = 1$, in Ex. 1, Art. 109,

$$\int \frac{x^3dx}{\sqrt{1-x^2}} = -\tfrac{1}{3}(1-x^2)^{\frac{1}{2}}(x^2+2).$$

Therefore $\int x^2 \text{ arc } \sin x dx = \frac{x^3}{3} \text{ arc } \sin x + \frac{1}{9}(1-x^2)^{\frac{1}{2}}(x^2+2)$.

PROBLEMS.

1. $\int \text{arc } \sin x dx = x \text{ arc } \sin x + (1-x^2)^{\frac{1}{2}}$.

2. $\int \frac{x^2 \text{ arc } \tan x dx}{1+x^2} = x \text{ arc } \tan x - \frac{1}{2}(\text{arc } \tan x)^2 - \frac{1}{2}\log(1+x^2)$.

3. $\int \frac{x^3 \text{ arc } \sin x dx}{\sqrt{1-x^2}} = -\frac{1}{3}(x^2+2)\sqrt{1-x^2} \text{ arc } \sin x + \frac{x^3}{9} + \frac{2}{3}x$.

4. $\int x \text{ arc } \cos x dx = \frac{1}{2}x^2 \text{ arc } \cos x - \frac{1}{4}x(1-x^2)^{\frac{1}{2}} + \frac{1}{4} \text{ arc } \sin x$.

ART. 117. INTEGRATION OF THE FORM $\int \frac{d\theta}{a + b\cos\theta}$

$$\int \frac{d\theta}{a+b\cos\theta} = \int \frac{d\theta}{a\left[\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\right] + b\left[\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right]}$$

$$=\int\frac{d\theta}{(a+b)\cos^2\left(\frac{\theta}{2}\right)+(a-b)\sin^2\left(\frac{\theta}{2}\right)}$$

$$=\int\frac{\sec^2\left(\frac{\theta}{2}\right)d\theta}{(a+b)+(a-b)\tan^2\left(\frac{\theta}{2}\right)}$$

$$=2\int\frac{d\tan\left(\frac{\theta}{2}\right)}{(a+b)+(a-b)\tan^2\left(\frac{\theta}{2}\right)}$$

$$=\frac{2}{\sqrt{a^2-b^2}}\arctan\left[\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}}\tan\left(\frac{\theta}{2}\right)\right],\text{ when } a>b. \quad (1)$$

If $a<b$, then

$$\int\frac{d\theta}{a+b\cos\theta}=2\int\frac{d\tan\left(\frac{\theta}{2}\right)}{(b+a)-(b-a)\tan^2\left(\frac{\theta}{2}\right)}$$

$$=\frac{1}{\sqrt{b^2-a^2}}\log\frac{\sqrt{b+a}+\sqrt{b-a}\tan\left(\frac{\theta}{2}\right)}{\sqrt{b+a}-\sqrt{b-a}\tan\left(\frac{\theta}{2}\right)}, \quad (2)$$

by Ex. 4, Art. 99.

In like manner $\int\frac{dx}{a+b\sin x}$ may be obtained.

PROBLEMS.

1. Find $\int\frac{d\theta}{2+\cos\theta}$.

Substituting $a=2$ and $b=1$ in (1) gives

$$\int\frac{d\theta}{2+\cos\theta}=\frac{2}{\sqrt{3}}\arctan\left(\sqrt{\tfrac{1}{3}}\tan\frac{\theta}{2}\right).$$

2. $\int\frac{d\theta}{3+5\cos\theta}=\frac{1}{4}\log\frac{\tan\frac{\theta}{2}+2}{\tan\frac{\theta}{2}-2}$.

3. $\int\frac{d\theta}{5-4\cos 2\theta}=\frac{1}{3}\arctan(3\tan x)$.

ART. 118. INTEGRATION BY SERIES.

When a given function cannot be integrated by any of the preceding methods, or when the integral obtained is too complicated in form for convenient use, recourse is had to the method of approximation called *integration by series*. By this method the function is developed into a series whose terms are integrated separately. If the resultant series is convergent, an approximate value of the integral is found by summing a finite number of terms.

PROBLEMS.

1. Find $\int x^2(1-x^2)^{\frac{1}{2}}\,dx$.

Expanding $(1-x^2)^{\frac{1}{2}}$ by the Binomial Theorem,

$$(1-x^2)^{\frac{1}{2}} = 1 - \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 - \tfrac{1}{16}x^6 \cdots.$$

Therefore $\int x^2(1-x^2)^{\frac{1}{2}}\,dx = \int x^2(1 - \tfrac{1}{2}x^6 - \tfrac{1}{8}x^4 - \tfrac{1}{16}x^6 \cdots)\,dx$

$$= \frac{x^3}{3} - \frac{x^5}{10} - \frac{x^7}{56} - \frac{x^9}{144} \cdots.$$

2. $\int \frac{dx}{1+x} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \cdots.$

3. $\int x^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}\,dx = \tfrac{2}{3}x^{\frac{3}{2}} - \tfrac{1}{7}x^{\frac{7}{2}} - \tfrac{1}{44}x^{\frac{11}{2}} - \tfrac{1}{120}x^{\frac{15}{2}} \cdots.$

4. $\int \frac{dx}{\sqrt{1+x^2}} = x - \frac{x^3}{2\cdot 3} + \frac{3x^5}{2\cdot 4\cdot 5} - \frac{3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} \cdots.$

5. $\int \frac{\sqrt{1-e^2x^2}}{\sqrt{1-x^2}}\,dx.$

$$(1-e^2x^2)^{\frac{1}{2}} = 1 - \tfrac{1}{2}e^2x^2 + \frac{1}{2\cdot 4}e^4x^4 - \frac{1\cdot 3}{2\cdot 4\cdot 6}e^6x^6 \cdots$$

Therefore

$$\int \frac{\sqrt{1-e^2x^2}}{\sqrt{1-x^2}}\,dx = \int\left(1 - \tfrac{1}{2}e^2x^2 + \frac{1}{2\cdot 4}e^4x^4 - \frac{1\cdot 3}{2\cdot 4\cdot 6}e^6x^6 \cdots\right)\frac{dx}{\sqrt{1-x^2}}$$

$$= \arcsin x + \tfrac{1}{2}e^2\left(\tfrac{1}{2}x\sqrt{1-x^2} - \tfrac{1}{2}\arcsin x\right)$$

$$- \frac{1}{2\cdot 4}e^4\left[\left(\tfrac{1}{4}x^3 + \tfrac{1}{2}\cdot\tfrac{3}{4}x\right)\sqrt{1-x^2} - \frac{1\cdot 3}{2\cdot 4}\arcsin x\right] \cdots.$$

6. $\int x^{-\frac{1}{2}}(x-1)^{\frac{2}{3}}\,dx = \tfrac{6}{7}x^{\frac{7}{6}} - 4x^{\frac{1}{6}} + \tfrac{2}{15}x^{-\frac{5}{6}} + \tfrac{8}{297}x^{-\frac{11}{6}} \cdots.$

CHAPTER XIX.

INTEGRATION AS A SUMMATION. AREAS AND LENGTHS OF PLANE CURVES.

ART. 119. INTEGRATION AS A SUMMATION.

The integration of a function may be regarded as the summation of a certain infinite series of infinitely small terms. The problem of finding the areas of plane curves furnishes a good illustration.*

For example, to find the area AP_1P_nN, included between the curve RS, the X-axis, and the ordinates AP_1 and NP_n.

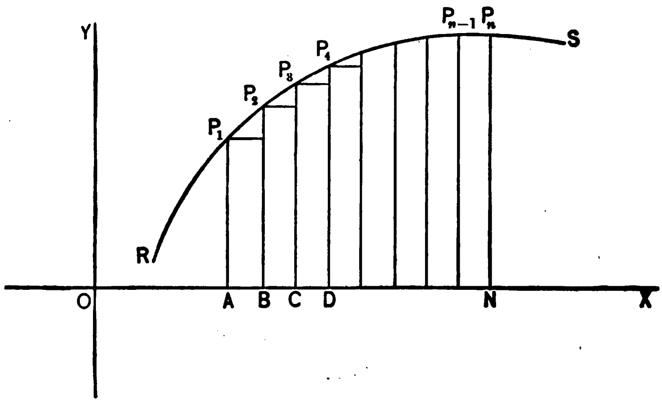

FIG. 32.

Let $y = f(x)$ be the equation of the curve. And let $OA = a$, $ON = b$, and divide AN into n equal parts each denoted by Δx, and erect ordinates at the points of division.

Then, area of rectangle $P_1B \quad = f(a)\Delta x$,

area of rectangle $P_2C \quad = f(a + \Delta x)\Delta x$,

* Newton's Lemma II., *Principia*, Lib. I., § 1.

$$\text{area of rectangle } P_3D \quad = f(a + 2\,\Delta x)\,\Delta x,$$

$$\cdots \qquad \cdots \qquad \cdots$$

and
$$\text{area of rectangle } P_{n-1}N = f(b - \Delta x)\,\Delta x.$$

Therefore, the sum of the n rectangles is

$$f(a)\Delta x + f(a + \Delta x)\Delta x + f(a + 2\,\Delta x)\Delta x + \cdots + f(b - \Delta x)\Delta x, \quad (1)$$

which may be represented by $\sum_a^b f(x)\Delta x$, in which $f(x)\Delta x$ represents each term of the series, x taking in succession the different values between a and b.

Now as Δx approaches zero, n increases indefinitely, and the limit of the sum of the rectangles is the required area AP_1P_nN.

When Δx becomes dx, the symbol $\sum$ is replaced by $\int$, and the expression for the area, which is the sum of an infinite number of infinitely small rectangles, becomes

$$\int_a^b f(x)\,dx = f(a)\,dx + f(a + dx)\,dx + f(a + 2\,dx)\,dx \cdots + f(b - dx)\,dx. \quad (2)$$

Assume
$$\int f(x)\,dx = \phi(x);$$

then
$$f(x)\,dx = d\phi(x) = \phi(x + dx) - \phi(x). \quad (3)$$

Substituting successively for x in (3) the values,

$$a,\ a + dx,\ a + 2\,dx, \cdots b - dx,$$

gives
$$\begin{aligned} f(a)\,dx \quad &= \phi(a + \quad dx) - \phi(a),\\ f(a + dx)\,dx \quad &= \phi(a + 2\,dx) - \phi(a + dx),\\ f(a + 2\,dx)\,dx &= \phi(a + 3\,dx) - \phi(a + 2\,dx),\\ \cdots \qquad &\quad \cdots \qquad \cdots\\ f(b - dx)\,dx \quad &= \phi(b) - \phi(b - dx). \end{aligned}$$

Adding these equations,

$$f(a)\,dx + f(a + dx)\,dx + \cdots f(b - dx)\,dx = \phi(b) - \phi(a),$$

or
$$\int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

Therefore the area is found by integrating $f(x)dx$, substituting b and a successively in the integral, and subtracting the latter result from the former.

PROBLEMS.

1. Find the area of the circle $x^2 + y^2 = r^2$.

Area of a quadrant $= \int_0^r f(x)\,dx = \int_0^r y\,dx = \int_0^r (r^2 - x^2)^{\frac{1}{2}} dx.$

By Ex. 4, Art. 109,

$$\int_0^r (r^2 - x^2)^{\frac{1}{2}} dx = \left[\frac{x(r^2 - x^2)^{\frac{1}{2}}}{2} + \frac{r^2}{2}\text{arc sin}\frac{x}{r}\right]_0^r$$
$$= \frac{\pi r^2}{4};$$

therefore the area of the whole circle is πr^2.

In order to obtain the area of the semi-segment $OABC$, Fig. 33, the superior limit of integration will be $OA = x$, and the inferior limit will be zero.

Therefore

$$\text{area } OABC = \int_0^x (r^2 - x^2)^{\frac{1}{2}} dx$$
$$= \frac{x(r^2 - x^2)^{\frac{1}{2}}}{2} + \frac{r^2}{2}\text{arc sin}\frac{x}{r}.$$

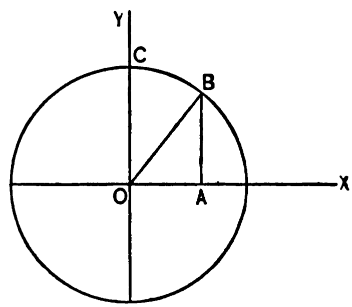

FIG. 33.

Evidently $\frac{x(r^2 - x^2)^{\frac{1}{2}}}{2} = \text{area of triangle } OAB,$

and $\frac{r^2}{2}\text{arc sin}\frac{x}{r} = \text{area of sector } OBC.$

2. Find the area between the curve $y^2 = 4x$, the axis of X, and the ordinate through the focus. *Ans.* $A = \frac{4}{3}$.

3. Find the area of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$. *Ans.* πab.

4. Find the area of the hyperbola $xy = 1$ between the limits $x = a$ and $x = 1$. *Ans.* $\text{Log}\, a$.

In this example it will be seen that the area of the hyperbola is the Naperian logarithm of the superior limit. For this reason Naperian logarithms are also called hyperbolic logarithms.

5. Find the area of the cycloid $x = r \operatorname{arc\,vers} \frac{y}{r} - \sqrt{2ry - y^2}$.

Here $$dx = \frac{y\,dy}{\sqrt{2ry - y^2}}.$$

Therefore $$A = 2\int_0^{2r} \frac{y^2 dy}{\sqrt{2ry - y^2}} = 3\pi r^2.$$

6. To find the area of $y(1 + x^2) = 1 - x^3$, between the curve and the axes, in the first quadrant.

The limits will be found to be $x = 1$ and $x = 0$.

Therefore
$$\begin{aligned} A &= \int_0^1 \frac{1 - x^3}{1 + x^2} dx \\ &= \int_0^1 \left[-x + \frac{x}{x^2 + 1} + \frac{1}{x^2 + 1} \right] dx \\ &= \left[-\frac{x^2}{2} + \tfrac{1}{2} \log(x^2 + 1) + \arctan x \right]_0^1 \\ &= .631972. \end{aligned}$$

7. Find the area included between $y^2 = 2px$ and $x^2 = 2py$.

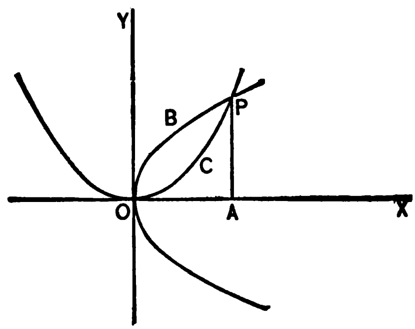

FIG. 34.

The two parabolas intersect at $(0, 0)$ and $(2p, 2p)$; hence the limits of integration are $2p$ and 0.

$$\text{Area } OBPA = \int_0^{2p} \sqrt{2px}\, dx.$$

$$\text{Area } OCPA = \int_0^{2p} \frac{x^2}{2p} dx.$$

Therefore $\qquad$ area $OBPC = \int_0^{2p}\left(\sqrt{2px} - \frac{x^2}{2p}\right)dx = \frac{4p^2}{3}.$

8. Find the area included between $y^2 = 2x$ and $y^2 = 4x - x^2$.

Ans. 0.475.

9. Find the entire area within the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Ans. $\frac{3\pi a^2}{8}$.

10. What is the area of a theoretical indicator diagram when the steam is cut off at half-stroke, if the law of expansion is $pv = 1$?

Ans. $1 + \log 2$.

Art. 120. Areas of Plane Curves in Polar Coördinates.

Referring to Fig. 35, it is required to find the area POP_n, included between any plane curve AB and two vectors OP and OP_n.

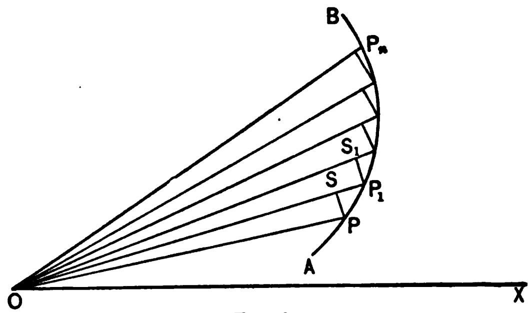

Fig. 35.

Let the vectorial angles POX and P_nOX be denoted respectively by β and α.

If the coördinates of any point P be (r, θ), then the coördinates of P_1 will be $(r + \Delta r, \theta + \Delta\theta)$.

The area of sector $POS = \frac{1}{2} r \cdot r\Delta\theta = \frac{1}{2} r^2\Delta\theta.$

Then the sum of all the sectors POS, P_1OS_1, etc., may be represented by $\sum_\beta^\alpha \frac{1}{2} r^2 \Delta\theta$; and as $\Delta\theta$ approaches zero, the limit of the sum of the sectors is the required area POP_n, which will be given by the expression

$$A = \frac{1}{2}\int_\beta^\alpha r^2 d\theta.$$

PROBLEMS.

1. Find the area of the logarithmic spiral $r = a^\theta$, between the limits r_2 and r_1.

Here $dr = a^\theta \log a\, d\theta$, and $d\theta = \dfrac{dr}{r \log a}$.

Hence
$$A = \tfrac{1}{2}\int_{r_1}^{r_2} r^2 d\theta = \frac{1}{2\log a}\int_{r_1}^{r_2} r\, dr = \left[\frac{r^2}{4\log a}\right]_{r_1}^{r_2}$$
$$= \frac{1}{4\log a}[r_2^2 - r_1^2].$$

2. Find the area described by one revolution of the radius vector of the spiral of Archimedes $r = a\theta$. *Ans.* $A = \frac{1}{2}\int_0^{2\pi} a^2\theta^2 d\theta = \dfrac{4\pi^3 a^2}{3}$.

3. Find the area of the lemniscate $r^2 = a^2 \cos 2\theta$. *Ans.* a^2.

4. Find the area of a loop of the curve $r = a\cos 2\theta$. *Ans.* $\frac{1}{8}\pi a^2$.

5. Find the entire area of the cardioid $r = a(1 - \cos\theta)$. *Ans.* $\dfrac{3\pi a^2}{2}$.

6. Find the area of a loop of the curve $r^2 \cos\theta = a^2 \sin 3\theta$.

Ans. $\dfrac{3a^2}{4} - \dfrac{a^2}{2}\log 2$.

Art. 121. Rectification of Plane Curves referred to Rectangular Axes.

By Art. 72,
$$ds = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx,$$
in which s represents the length of the arc.

Therefore
$$s = \int_a^b \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx; \tag{1}$$
the limits of integration being the limiting values of x.

The process of finding the length of an arc of a curve is called the rectification of the curve.

If y be considered the independent variable, the formula is
$$s = \int_c^d \left[1 + \left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}} dy, \tag{2}$$
in which d and c are the limiting values of y.

If the arc PR, in Fig. 36, is to be rectified, the value of the first derivative is found from the equation of the curve, and substituted in

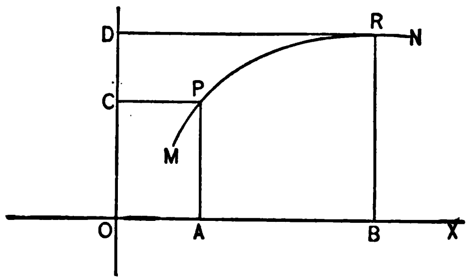

FIG. 36.

Formula (1) or (2). If Formula (1) is used, the limits b and a are OB and OA respectively; if Formula (2) is used, the limits d and c are OD and OC respectively.

PROBLEMS.

1. Rectify the parabola $y^2 = 2px$.

Here $$\frac{dy}{dx} = \frac{p}{y};$$

hence $$dx = \frac{y\,dy}{p}.$$

Therefore $$s = \int\left(1 + \frac{p^2}{y^2}\right)^{\frac{1}{2}} dx = \frac{1}{p}\int (p^2 + y^2)^{\frac{1}{2}} dy$$

$$= \frac{y\sqrt{p^2+y^2}}{2p} + \frac{p}{2}\log\left(y + \sqrt{p^2+y^2}\right) + C. \qquad (1)$$

Here the value of the constant C may be determined by the first method of Art. 34. If the arc is estimated from the origin, then $S = 0$ when $y = 0$, and these values substituted in (1) give

$$0 = \frac{p}{2}\log p + C;$$

hence $$C = \frac{-p}{2}\log p.$$

Therefore $$s = \frac{y\sqrt{p^2+y^2}}{2p} + \frac{p}{2}\log\left(\frac{y + \sqrt{p^2+y^2}}{p}\right),$$

which is the length from the vertex to the point which has the ordinate y.

Or if the limits of integration are known, for instance, if the length from the vertex to an extremity of the latus-rectum is required, then the limits are p and 0, and

$$s=\left[y\,\frac{\sqrt{p^2+y^2}}{2p}+\frac{p}{2}\log\left(y+\sqrt{p^2+y^2}\right)\right]_0^p=\tfrac{1}{2}p\sqrt{2}+\frac{p}{2}\log\left(1+\sqrt{2}\right).$$

2. * Rectify the semi-cubical parabola $y^2=ax^3$.

Ans. $\dfrac{8}{27a}\left(1+\tfrac{9}{4}ax\right)^{\frac{3}{2}}-\dfrac{8}{27a}.$

3. Rectify the curve whose equation is $y^2=\dfrac{x^3}{4}$, and determine the length of the curve from the origin to the point whose ordinate is 10.

Ans. 19.0248.

4. Rectify the circle $x^2+y^2=r^2$.

Here $$s=4\int_0^r\left(1+\frac{x^2}{y^2}\right)^{\frac{1}{2}}dx=4r\int_0^r\frac{dx}{\sqrt{r^2-x^2}}=2\pi r.$$

But as the result is in circular measure, the circle is a non-rectifiable curve.

An approximate result may be obtained by a series.

$$s=4r\int_0^r\frac{dx}{\sqrt{r^2-x^2}}=4r\left[\frac{x}{r}+\frac{x^3}{2\cdot 3r^3}+\frac{1\cdot 3x^5}{2\cdot 4\cdot 5r^5}+\cdots\right]_0^r$$

$$=4r\left[1+\frac{1}{2\cdot 3}+\frac{1\cdot 3}{2\cdot 4\cdot 5}+\frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 7}+\cdots\right].$$

Therefore

$$\pi=2\left[1+\frac{1}{2\cdot 3}+\frac{1\cdot 3}{2\cdot 4\cdot 5}+\frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 7}+\cdots\right].$$

From this equation the approximate value of π can be determined with any required degree of accuracy by taking a sufficient number of terms.

5. Rectify the ellipse $y^2=(1-e^2)(a^2-x^2)$.

Here $$\frac{dy}{dx}=-(1-e^2)\frac{x}{y}=-\frac{x\sqrt{1-e^2}}{\sqrt{a^2-x^2}}.$$

* The semi-cubical parabola was the first curve whose rectification was effected algebraically. (Neil, in 1657, *Phil. Trans.*, 1673.)

Hence $$s = 4\int_0^a \sqrt{\frac{a^2 - e^2x^2}{a^2 - x^2}}\,dx = 4\int_0^a \frac{dx}{\sqrt{a^2 - x^2}}(a^2 - e^2x^2)^{\frac{1}{2}}$$

$$= 4\int_0^a \frac{dx}{\sqrt{a^2 - x^2}}\left(a - \frac{e^2x^2}{2a} - \frac{e^4x^4}{2\cdot 4a^3} - \cdots\right)$$

$$= 2\pi a\left(1 - \frac{e^2}{2^2} - \frac{3e^4}{2^2\cdot 4^2} - \cdots\right).$$

6. Rectify the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Ans. $S = \frac{3}{2}a^{\frac{1}{3}}x^{\frac{2}{3}}$; the entire curve $= 6a$.

7. Rectify the cycloid $x = r \text{ arc vers } \frac{y}{r} - \sqrt{2ry - y^2}$.

Here $$\frac{dx}{dy} = \frac{y}{\sqrt{2ry - y^2}}.$$

Therefore $$s = 2\int_0^{2r}\left(\frac{2r}{2r - y}\right)^{\frac{1}{2}} dy = 8r.$$

ART. 122. RECTIFICATION OF CURVES IN POLAR COÖRDINATES.

By Art. 73 (2),

$$ds = \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{1}{2}} d\theta.$$

Therefore $$s = \int_\alpha^\beta \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{1}{2}} d\theta.$$

PROBLEMS.

1. To find the length of the cardioid $r = a(1 + \cos\theta)$.

Here $$\frac{dr}{d\theta} = -a\sin\theta.$$

Therefore $$s = 2\int_0^\pi [a^2(1 + \cos\theta)^2 + a^2\sin^2\theta]^{\frac{1}{2}} d\theta$$

$$= 2a\int_0^\pi (2 + 2\cos\theta)^{\frac{1}{2}} d\theta = 4a\int_0^\pi \cos\frac{\theta}{2}\,d\theta$$

$$= \left[8a\sin\frac{\theta}{2}\right]_0^\pi = 8a.$$

2. Rectify the spiral of Archimedes $r = a\theta$.

$$Ans.\ \frac{r(a^2+r^2)^{\frac{1}{2}}}{2a} + \frac{a}{2}\log\left(\frac{r+\sqrt{a^2+r^2}}{a}\right).$$

3. Rectify the logarithmic spiral $\log r = \theta$ between the limits r_1 and r_0. *Ans.* $(1+m^2)^{\frac{1}{2}}(r_1 - r_0)$.

4. Rectify the curve $r = a\sin^3\frac{\theta}{3}$. *Ans.* $\frac{3\pi a}{2}$.

Art. 123. The Common Catenary.

The common catenary is the curve assumed by a flexible cord of uniform thickness and density, fastened at two points, hanging freely and acted upon only by the force of gravity.

As the cord is regarded as perfectly flexible, the only force acting at any point of the cord is a pull in the direction of the cord at that point, which is called tension and is a function of the coördinates of that point.

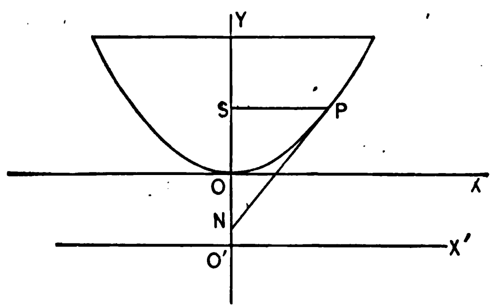

Fig. 37.

In Fig. 37, let O, the lowest point of the curve, be the origin, and let the horizontal line through O be the X-axis, and the vertical OY be the Y-axis. Let (x, y) be any point P on the curve, s the length of OP, and c the length of the cord whose weight is equal to the tension at O.

If the weight of the unit of length be taken as the unit of weight, the length s will represent the weight of the arc OP, and the length c will represent the tension at O.

Then the arc OP may be regarded as a rigid body in equilibrium under three forces: the tension at P in the direction of the tangent, the horizontal tension c at the origin, and the weight s acting vertically downward.

Draw PN tangent to the curve, and PS parallel to OX at P. Then by the triangle of forces, the sides of the triangle PSN will represent the three forces acting on the arc OP.

Therefore $$\frac{NS}{SP} = \frac{\text{weight of } OP}{\text{tension at } O} = \frac{s}{c};$$

hence $$\frac{dy}{dx} = \frac{s}{c}. \tag{1}$$

Differentiating (1), substituting the value of ds, and reducing,

$$\frac{d\left(\frac{dy}{dx}\right)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \frac{dx}{c}. \tag{2}$$

Integrating (2) and noticing that when $x = 0$, $\frac{dy}{dx} = 0$,

$$\log\left[\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}\right] = \frac{x}{c},$$

or $$\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2} = e^{\frac{x}{c}};$$

whence $$\frac{dy}{dx} = \tfrac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right). \tag{3}$$

Integrating (3), and noticing that $x = 0$ when $y = 0$,

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) - c, \tag{4}$$

which is the required equation.

Removing the origin to the point O', which is at a distance c below O, the equation becomes

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right). \tag{5}$$

In order to rectify the catenary, (5) is differentiated, giving

$$\frac{dy}{dx}=\frac{1}{2}\left(e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right), \text{ as in (3)},$$

from which is obtained

$$ds=\frac{1}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)dx.$$

Therefore
$$s=\frac{1}{2}\int_0^x\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)dx=\frac{c}{2}\left(e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right).$$

PROBLEM.

What is the curve in which the cables of a suspension bridge hang?

Ans. A parabola.

CHAPTER XX.

SURFACES AND VOLUMES OF SOLIDS.

ART. 124. SURFACES AND VOLUMES OF SOLIDS OF REVOLUTION.

1st. *Surfaces.* In Fig. 38, let the plane curve MN revolve about the X-axis. Let M be a fixed point and P any other point of the curve whose coördinates are (x, y).

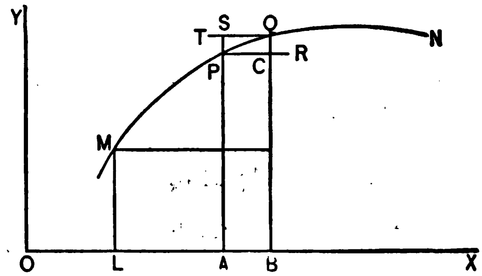

FIG. 38.

Assume $MP = s$ and $PQ = \Delta s$, then the coördinates of Q are $(x + \Delta x, y + \Delta y)$. Let S represent the area of the surface generated by the revolution of MP, and ΔS the surface generated by PQ. Draw PR and QT, each equal in length to Δs, and parallel to OX. In the revolution PR generates the convex surface of a cylinder whose area is $2\pi y \Delta s$, and QT generates the convex surface of a cylinder whose area is $2\pi(y + \Delta y)\Delta s$. Obviously the area of the surface generated by PQ lies between the areas of the cylindrical surfaces.

Hence $$2\pi y \Delta s < \Delta S < 2\pi(y + \Delta y)\Delta s.$$

Therefore, as Δs approaches zero,

$$dS = 2\pi y\, ds,$$

and
$$S=\int 2\pi y\, ds \qquad (1)$$

$$=2\pi\int y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx. \qquad (2)$$

In like manner for the surface generated by revolving the curve about the Y-axis,

$$S=2\pi\int x\, ds=2\pi\int x\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}} dy. \qquad (3)$$

The surface of a zone, included between two planes perpendicular to the X-axis and corresponding to the abscissas b and a, is

$$S=2\pi\int_a^b y\, ds. \qquad (4)$$

2d. *Volumes.* Let V denote the volume generated by the surface included by the curve MP, the ordinates ML and PA, and the X-axis.

Let ΔV represent the volume generated by $APQB$.

The volume of the cylinder generated by $APCB$ is $\pi y^2\Delta x$, and the volume of the cylinder generated by $ASQB$ is $\pi(y+\Delta y)^2\Delta x$.

Obviously, $\pi y^2\Delta x<\Delta V<\pi(y+\Delta y)^2\Delta x.$

Therefore, as Δx approaches zero,

$$dV=\pi y^2 dx,$$

and
$$V=\pi\int y^2 dx. \qquad (5)$$

In like manner, for the volume generated by revolving the curve about the Y-axis,

$$V=\pi\int x^2 dy. \qquad (6)$$

PROBLEMS.

1. Find the surface of the sphere generated by revolving the circle $x^2+y^2=r^2$ about a diameter.

Here $y=(r^2-x^2)^{\frac{1}{2}}$ and $\frac{dy}{dx}=-\frac{x}{y}$.

Therefore $S = 2\pi \int_{-r}^{r} y\left(1 + \frac{x^2}{y^2}\right)^{\frac{1}{2}} dx = 2\pi r \int_{-r}^{r} dx = 4\pi r^2.$

2. Find the volume generated by revolving the parabola $y^2 = 2px$ about the X-axis.

$$V = \pi \int_0^x 2\,px\,dx = p\pi x^2 = \tfrac{1}{2}\pi y^2 \cdot x.$$

3. Find the volume of the cone generated by revolving $y = x \tan a$, when a is the semi-vertical angle of the cone.

Ans. $V = \frac{1}{3}$ volume of circumscribing cylinder.

4. Find the volume of the sphere generated by revolving $x^2+y^2=r^2$ about the X-axis, and also the volume of a spherical segment between two parallel planes at distances b and a from the centre.

Ans. $\frac{4}{3}\pi r^3$ and $\pi\left[r^2(b-a) - \frac{1}{3}(b^3 - c^3)\right].$

5. Find the surface and volume of the prolate spheroid generated by revolving $y^2 = (1-e^2)(a^2-x^2)$ about the X-axis.

Ans. $S = 2\pi b^2 + \frac{2\pi ab}{e}\text{arc sin}\, e$ and $V = \frac{4\pi ab^2}{3}.$

6. Find the surface and volume of the right circular cone, generated by revolving the line joining the origin with the point (a, b) about the X-axis.

Ans. $S = \pi b\sqrt{a^2+b^2}$ and $V = \frac{\pi ab^2}{3}.$

7. Find the surface generated by the cycloid

$$y = r\,\text{arc vers}\,\frac{x}{r} + \sqrt{2\,rx - x^2},$$

when it revolves about its axis. *Ans.* $8\pi r^2(\pi - \frac{4}{3}).$

8. Find the volume generated by the cycloid

$$x = r\,\text{arc vers}\,\frac{y}{r} - \sqrt{2\,ry - y^2},$$

when it revolves about its base. *Ans.* $5\pi^2 r^3.$

9. Find the surface and volume of the annular torus, generated by revolving the circle $x^2 + (y - 5'')^2 = 4''$, about the X-axis.

Ans. $S = 394.79$ sq. in., and $V = 394.79$ cu. in.

ART. 125. SURFACES BY DOUBLE INTEGRATION.

In Fig. 39, let (x, y, z) and $(x + dx, y + dy, z + dz)$ be the coördinates of two consecutive points P and E on the given surface whose equation is known. Through P and E pass planes parallel to the planes XZ and YZ. These planes will intercept an element PE of the curved surface, which is projected on the XY-plane in $RS = dx\,dy$.

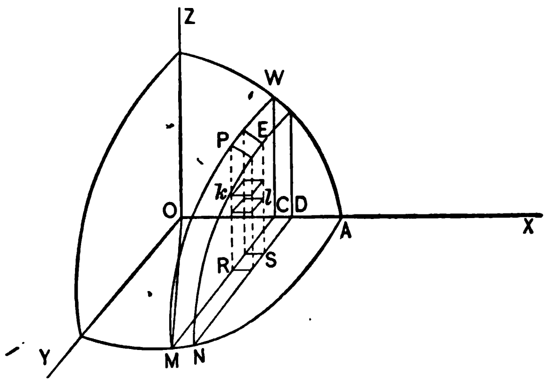

FIG. 39.

Let S represent the required area, and dS the area of the element PE.

The area of RS is evidently equal to the area of PE, multiplied by the cosine of the angle which PE makes with XY.

Representing this angle by γ,

$$\text{area } PE \cdot \cos\gamma = dx\,dy;$$

hence

$$\text{area } PE = dx\,dy \cdot \sec\gamma.$$

By the aid of analytical geometry of three dimensions,*

$$\sec\gamma = \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}},$$

in which $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are partial derivatives from the equation of the given surface.

* See Appendix; Note A.

Therefore area $PE = dS = \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}} dx\,dy,$

and $$S = \int\int\left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}} dx\,dy.$$

The effect of the y-integration, x remaining constant, will be to give the sum of all the elements similar to PE from W to M; hence the limits of the y-integration will be $y = CM = \sqrt{OM^2 - x^2}$ and $y = 0$.

The effect of the subsequent x-integration will be to give the sum of all the strips similar to WMN forming the given surface; hence the limits of the second integration are $x = OA$ and $x = 0$.

ART. 126. VOLUMES BY TRIPLE INTEGRATION.

The given volume is supposed to be divided into elementary rectangular parallelopipeds by planes parallel to the three coördinate planes; such an element of volume is represented by kl in Fig. 39.

The volume of such an elementary parallelopiped is $dx\,dy\,dz$; hence the whole volume is

$$V = \int\int\int dx\,dy\,dz. \tag{1}$$

The limits of integration are obtained from the equation of the bounding surface, being so chosen as to embrace the entire volume.

If the volume included between the three coördinate planes and the curved surface is required, the limits are found as follows:

The effect of the z-integration is to sum all the elemental parallelopipeds in the prism PS; hence the limits of the first integration are $PR = z = f(x, y)$ and $z = 0$. The effect of the y-integration is to sum all the elemental prisms in the slice WN; hence the limits for the second integration are $CM = y = f(x)$ and $y = 0$. The effect of the x-integration is to sum all the elemental slices composing the whole volume; hence the limits of the third integration are $x = OA$ and $x = 0$.

For example, to find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, cut off by the coördinate planes.

Here the limits of the z-integration are $c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$ and 0, the

limits of the y-integration are $b\sqrt{1-\frac{x^2}{a^2}}$ and 0, and the limits of the x-integration are a and 0.

Substituting $z_1 = c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$, and $y_1 = b\sqrt{1-\frac{x^2}{a^2}}$ in the formula, gives for the whole ellipsoid

$$V = 8\int_0^a\int_0^{y_1}\int_0^{z_1} dx\,dy\,dz.$$

Integrating on the hypothesis that z is the only variable,

$$V = 8c\int_0^a\int_0^{y_1}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}dx\,dy.$$

Integrating again, now on the hypothesis that y is the only variable,

$$V = \frac{8c}{b}\int_0^a\int_0^{y_1}(y_1^2-y^2)^{\frac{1}{2}}\,dx\,dy = \frac{8c}{b}\int_0^a\left[\frac{y}{2}(y_1^2-y^2)^{\frac{1}{2}}+\frac{y_1^2}{2}\text{ arc sin }\frac{y}{y_1}\right]_0^{y_1}dx$$

$$= \frac{8c}{b}\int_0^a\frac{y_1^2}{2}\frac{\pi}{2}\,dx.$$

Integrating finally with respect to x, gives

$$V = \frac{2\pi cb}{a^2}\int_0^a(a^2-x^2)\,dx = \tfrac{4}{3}\pi abc.$$

PROBLEMS.

1. Find the surface of the sphere $x^2+y^2+z^2=a^2$.

Here $$\frac{\partial z}{\partial x} = -\frac{x}{z}, \text{ and } \frac{\partial z}{\partial y} = -\frac{y}{z}.$$

Therefore $$S = \int\int\left(1+\frac{x^2}{z^2}+\frac{y^2}{z^2}\right)^{\frac{1}{2}}dx\,dy = \int\int\frac{a\,dx\,dy}{\sqrt{a^2-x^2-y^2}}.$$

Integrating with respect to y, between $y=\sqrt{a^2-x^2}$ and $y=0$,

$$S = \int\int_0^{\sqrt{a^2-x^2}}\frac{a\,dx\,dy}{\sqrt{a^2-x^2-y^2}} = \int\left[a\,dx\text{ arc sin }\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}}$$

$$= \int\frac{\pi}{2}a\,dx.$$

Integrating with respect to x, between $x = a$ and $x = 0$,

$$S = \int_0^a \frac{\pi}{2} a\, dx = \frac{\pi a^2}{2},$$

which is the area of one-eighth of the surface of the sphere.

2. A sphere $x^2 + y^2 + z^2 = a^2$ is cut by a right circular cylinder $y^2 = ax - x^2$. Find the area of the surface of the sphere intercepted by the cylinder. *Ans.* $2a^2(\pi - 2)$.

3. Find the surface intercepted by two right circular cylinders $x^2 + z^2 = a^2$ and $x^2 + y^2 = a^2$. *Ans.* $8a^2$.

4. Find the volume of a right elliptic cylinder whose axis coincides with the X-axis and whose altitude $= 2a$, the equation of the base being $c^2y^2 + b^2z^2 = b^2c^2$. *Ans.* $2\pi abc$.

5. Find the volume of the solid contained between the paraboloid of revolution $x^2 + y^2 = 2z$, the cylinder $x^2 + y^2 = 4x$, and the plane $z = 0$.

$$\textit{Ans. } 2\int_0^4 \int_0^{\sqrt{4x - x^2}} \int_0^{\frac{x^2+y^2}{2}} dx\, dy\, dz = 37.699+.$$

6. Find the volume of the solid cut from the cylinder $x^2 + y^2 = a^2$ by the planes $z = 0$ and $z = x \tan a$. *Ans.* $\frac{2}{3} a^3 \tan a$.

7. Find the entire volume bounded by the surface $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = \sqrt[3]{25}$. *Ans.* 44.88.

CHAPTER XXI.

CENTRE OF MASS. MOMENT OF INERTIA. PROPERTIES OF GULDIN.

ART. 127. DEFINITIONS.

The definitions of this article are taken from Mechanics and are here assumed without investigation.

The moment of any force with respect to an axis perpendicular to its line of direction is the product of the magnitude of the force by the perpendicular distance from its line of direction to the axis. The moment of a force with respect to a plane parallel to its line of direction is the product of the force by the perpendicular distance from its line of direction to the plane.

The force exerted by gravity on any body is proportional to the mass of the body, and hence the mass of the body may be taken as the measure of the force exerted on it by gravity.

The centre of mass of a body is that point so situated that the force of gravity produces no tendency in the body to rotate about any line passing through the point; hence it may be regarded as the point at which the whole weight of the body acts. The centre of mass is sometimes called centre of gravity and centre of inertia.

The moment of inertia of a body with reference to a straight line, or plane, is the sum of the products obtained by multiplying the mass of each element of the body by the square of its distance from the line or plane.

Points, lines and surfaces, as here considered, are supposed to be material bodies. Lines, surfaces and solids are regarded as being composed of an infinitely large number of indefinitely small particles. The weight of a body is the resultant of the weights of all of its elemental particles acting in vertical lines, and the resultant of this system of parallel forces passes through the centre of mass.

ART. 128. GENERAL FORMULAS FOR CENTRE OF MASS.

Assume a system of rectangular coördinate axes, retaining a fixed position with reference to the body, the plane XY being horizontal. Let a small particle of mass at any point (x, y, z) be represented by Δm. Then the force exerted by gravity on Δm is measured by Δm in a direction parallel to the Z-axis.

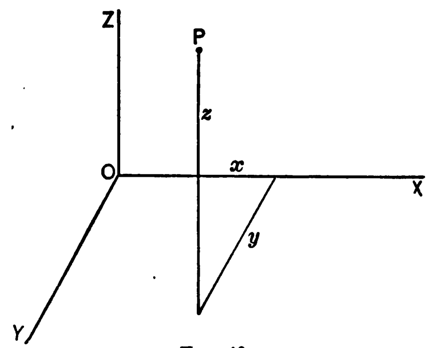

FIG. 40.

If the mass of Δm were concentrated at the point (x, y, z), the moment of the force exerted on Δm with respect to the plane YZ would be $x\Delta m$; and the sum of the moments of all the elements of the body with reference to this plane would be $\Sigma x\Delta m$.

The resultant force of gravity is $\Sigma\Delta m$, and if the coördinates of the centre of mass be represented by $(\bar{x}, \bar{y}, \bar{z})$, as the centre of mass is the point through which the resultant passes, $\bar{x}\Sigma\Delta m$ will be the moment of the resultant with reference to the plane YZ. But by the principle of moments, the moment of the resultant of any number of forces is equal to the algebraic sum of the moments of the forces.

Hence,
$$\bar{x}\Sigma\Delta m = \Sigma x\Delta m.$$

If now Δm diminishes indefinitely,
$$\bar{x}\int dm = \int x\,dm.$$

Therefore
$$\bar{x} = \frac{\int x\,dm}{\int dm}. \qquad (1)$$

Similarly,
$$\bar{y}=\frac{\int y\,dm}{\int dm},\tag{2}$$

and
$$\bar{z}=\frac{\int z\,dm}{\int dm}.\tag{3}$$

The mass of any homogeneous body is the product of its volume by its density. If k represents the constant density and dv the element of volume, then $k\,dv=dm$, and (1), (2) and (3) become

$$\bar{x}=\frac{\int x\,dv}{\int dv},\tag{4}$$

$$\bar{y}=\frac{\int y\,dv}{\int dv},\tag{5}$$

and
$$\bar{z}=\frac{\int z\,dv}{\int dv},\tag{6}$$

If the body is a material line in the form of the arc of any curve, and if ds is the length of an element of the curve, Formulas (4), (5) and (6) become

$$\bar{x}=\frac{\int x\,ds}{\int ds},\tag{7}$$

$$\bar{y}=\frac{\int y\,ds}{\int ds},\tag{8}$$

$$\bar{z}=\frac{\int z\,ds}{\int ds}.\tag{9}$$

If the curve is a plane curve, it may be taken in the plane XY, in which case $\bar{z}$ will be zero.

PROBLEMS.

1. Find the centre of mass of an arc of a circle, taking the diameter bisecting the arc as the X-axis and the left vertex as the origin.

In Fig. 41, let AOB be the arc.

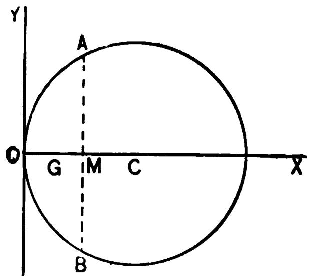

FIG. 41.

The equation of the circle is $y^2 = 2ax - x^2$;

hence $$dy = \frac{(a-x)\,dx}{\sqrt{2ax - x^2}}.$$

$$ds = \sqrt{dx^2 + dy^2} = \frac{a\,dx}{\sqrt{2ax - x^2}}.$$

Therefore $$\bar{x} = \frac{\int x\,ds}{\int ds} = \frac{a}{s}\int_0^x \frac{x\,dx}{\sqrt{2ax - x^2}} = \frac{a}{s}(-\sqrt{2ax - x^2} + s) = a - \frac{ay}{s}.$$

2. Find the centre of mass of an arc of the hypocycloid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

between two successive cusps:

Here $$dy = -\left(\frac{x}{y}\right)^{-\frac{1}{3}} dx;$$

hence $$ds = \sqrt{dx^2 + dy^2} = y^{\frac{1}{3}}\sqrt{x^{-\frac{2}{3}} + y^{-\frac{2}{3}}}\,dx$$

$$= \sqrt{a^{\frac{2}{3}} - x^{\frac{2}{3}}}\sqrt{x^{-\frac{2}{3}} + \frac{1}{a^{\frac{2}{3}} - x^{\frac{2}{3}}}}\,dx = \left(\frac{a}{x}\right)^{\frac{1}{3}} dx.$$

Therefore $$OM = \bar{x} = \frac{\int_0^a x\left(\frac{a}{x}\right)^{\frac{1}{3}} dx}{\int_0^a \left(\frac{a}{x}\right)^{\frac{1}{3}} dx} = \tfrac{2}{5}a.$$

Similarly, $$MG = \bar{y} = \tfrac{2}{5}a.$$

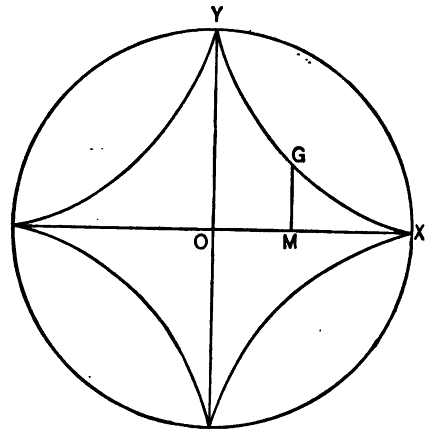

Fig. 42.

3. Find the centre of mass of the arc of a semi-cycloid.

Ans. $\bar{x} = (\pi - \tfrac{4}{3})a, \ \bar{y} = -\tfrac{2}{3}a.$

Art. 129. Centre of Mass of Plane Surfaces.*

If rectangular coördinates are used,

$$dv = dA = dx\,dy,$$

and Formulas (4) and (5) of Art. 128 become

$$\bar{x} = \frac{\int\int x\,dx\,dy}{\int\int dx\,dy}, \tag{1}$$

and $$\bar{y} = \frac{\int\int y\,dx\,dy}{\int\int dx\,dy}. \tag{2}$$

* The centre of mass of a plane area is sometimes called the centroid of the area.

PROBLEMS.

1. Find the centre of mass of the area included between the parabola $y^2 = 2\,px$ and the double ordinate whose abscissa is a.

$$\bar{x} = \frac{\int_0^a \int_{-y}^{y} x\,dx\,dy}{\int_0^a \int_{-y}^{y} dx\,dy} = \frac{\int_0^a xy\,dx}{\int_0^a y\,dx}$$

$$= \frac{\int_0^a \sqrt{2p}\,x^{\frac{3}{2}}dx}{\int_0^a \sqrt{2p}\,x^{\frac{1}{2}}dx} = \tfrac{3}{5}\,a.$$

2. Find the centre of mass of the semicircle $x^2 + y^2 = r^2$ on the right of the Y-axis.

$$\bar{x} = \frac{\int_0^r \int_{-y}^{y} x\,dx\,dy}{\int_0^r \int_{-y}^{y} dx\,dy} = \frac{\int_0^r xy\,dx}{\int_0^r y\,dx}$$

$$= \frac{\int_0^r x\sqrt{r^2 - x^2}dx}{\int_0^r \sqrt{r^2 - x^2}dx} = \frac{4}{3}\frac{r}{\pi}.$$

$$\bar{y} = 0.$$

3. Find the centre of mass of an elliptic quadrant whose equation is $y = \frac{b}{a}\sqrt{a^2 - x^2}$. *Ans.* $\bar{x} = \frac{4\,a}{3\,\pi},\ \bar{y} = \frac{4\,b}{3\,\pi}.$

4. In Fig. 43, ABD is a segment of a parabola cut off by an ordinate, and BE is parallel to Ax.

1st. Determine the distance of the centre of mass of ABD from Ax. *Ans.* $\frac{3\,y}{8}.$

2d. Determine the centre of mass of ABE. *Ans.* $\left(\frac{3\,x}{10}, \frac{3\,y}{4}\right).$

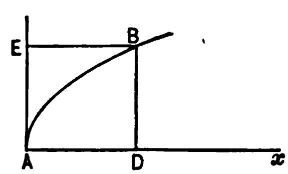

FIG. 43.

5. Find the centre of mass of the cycloid. *Ans.* $\bar{x} = \pi r,\ \bar{y} = \frac{5}{6}\,r.$

6. Find the centre of mass between $y^m = x^n$ and $y^n = x^m$.

$$\textit{Ans.}\ \bar{x} = \bar{y} = \frac{(m+n)^2}{(m+2n)(2m+n)}.$$

ART. 130. CENTRE OF MASS OF SURFACES OF REVOLUTION.

If a curve in a plane with the X-axis be revolved about this axis, then

$$dv = 2\pi y\,ds;$$

hence, by Art. 128 (4),

$$\bar{x} = \frac{\int 2\pi xy\,ds}{\int 2\pi y\,ds} = \frac{\int xy\,ds}{\int y\,ds}.$$

PROBLEMS.

1. Find the centre of mass of the convex surface of a right cone, generated by the line $y = ax$.

Here $$ds = \sqrt{dx^2 + dy^2} = \sqrt{a^2+1}\,dx;$$

hence $$\bar{x} = \frac{\int ax^2\sqrt{a^2+1}\,dx}{\int ax\sqrt{a^2+1}\,dx} = \tfrac{2}{3}x.$$

2. Find the centre of mass of the surface generated by the revolution of a semi-cycloid $x = a\,\text{vers}^{-1}\frac{y}{a} - \sqrt{2ay - y^2}$ about its base.

Here $$ds = \frac{\sqrt{2a}\,dy}{\sqrt{2a-y}};$$

hence $$\bar{x} = \frac{\int_0^{2a} \frac{xy\,dy}{\sqrt{2a-y}}}{\int_0^{2a} \frac{y\,dy}{\sqrt{2a-y}}} = \tfrac{4}{15}a.$$

3. Find the centre of mass of the convex surface of a hemisphere whose radius is equal to 10. *Ans.* $\bar{x} = 5$.

Art. 131. Centre of Mass of Solids of Revolution.

If a solid be generated by the revolution of a plane curve about the X-axis, then

$$dv = 2\,\pi y\,dy\,dx;$$

hence, by Art. 128 (4),

$$\bar{x} = \frac{\int\int xy\,dx\,dy}{\int\int y\,dx\,dy}.$$

PROBLEMS.

1. Find the centre of mass of a right circular cone, whose convex surface is generated by revolving $y = ax$ about the X-axis.

$$\bar{x} = \frac{\int_0^x\int_0^{ax} xy\,dx\,dy}{\int_0^x\int_0^{ax} y\,dx\,dy}$$

$$= \frac{\int_0^x \frac{a^2x^3}{2}dx}{\int_0^x \frac{a^2x^2}{2}dx} = \tfrac{3}{4}x.$$

2. Find the centre of mass of a paraboloid generated by $y^2 = 4\,ax$.

$$\bar{x} = \frac{\int_0^x\int_0^{2\sqrt{ax}} xy\,dx\,dy}{\int_0^x\int_0^{2\sqrt{ax}} y\,dx\,dy}$$

$$= \frac{\int_0^x 2\,ax^2dx}{\int_0^x 2\,ax\,dx} = \tfrac{2}{3}x.$$

3. Find the centre of mass of a hemispheroid generated by

$$y^2 = \frac{b^2}{a^2}(2\,ax - x^2). \qquad \textit{Ans. } \tfrac{5}{8}\,a.$$

ART. 132. MOMENTS OF INERTIA OF SURFACES.

In Fig. 44, the curves AB and CD and the ordinates LN and PM intercept a plane surface $PLSR$, whose moment of inertia is required.

The surface is supposed to be divided into rectangular elements by lines parallel to the coördinate axes.

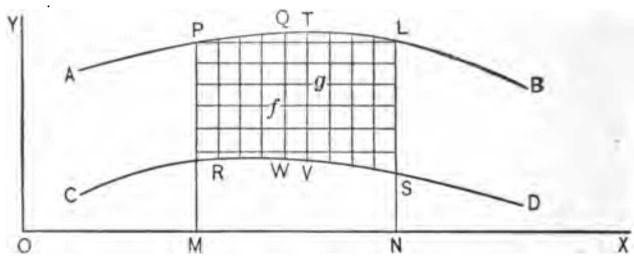

FIG. 44.

Let (x, y) be the coördinates of any point as f, then $(x + dx, y + dy)$ will be the coördinates of g, and $dx\,dy$ will be the area of the element fg.

The moment of fg about $X = y^2 dx\,dy$.

Let the equations of AB and CD be $y = f(x)$ and $y = \phi(x)$ respectively; and let $ON = b$ and $OM = a$.

If x be regarded as constant, while y varies from $\phi(x)$ to $f(x)$, the integration will give the moment of the vertical strip $WQTV$.

Then in the second integration, x varying from a to b, the sum of the moments of all the strips composing the area $PRSL$ will be given.

Representing the moment of inertia by M. I.,

$$\text{M. I.} = \int_a^b \int_{\phi(x)}^{f(x)} y^2 dx\,dy.$$

PROBLEMS.

1. Find the moment of inertia of a circle about its diameter.

$$\text{M. I.} = \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} y^2 dx\,dy$$

$$= \tfrac{2}{3} \int_{-r}^{r} (r^2 - x^2)^{\frac{3}{2}} dx = \tfrac{1}{4}\pi r^4.$$

2. Find the moment of inertia of a rectangle about an axis through its centre parallel to one of its sides.

Let $2b$ and $2d$ denote the width and length respectively, the axis being parallel to b; then

$$\text{M. I.} = \int_{-b}^{b}\int_{-d}^{d} y^2 dx\, dy = \tfrac{4}{3}\, bd^3.$$

3. Find the moment of inertia of an isosceles triangle about an axis which passes through its vertex and bisects its base.

Let $a =$ the altitude and $2b =$ base, and take the origin at the vertex and the axis of moments as the X-axis; then

$$\text{M. I.} = \int_0^a \int_{-\frac{b}{a}x}^{\frac{b}{a}x} y^2 dx\, dy = \tfrac{1}{6}\, ab^3.$$

Art. 133. Guldin's Theorems.*

I. Let a plane curve in the same plane with the X-axis revolve about the X-axis.

The ordinate of the centre of mass is

$$\bar{y} = \frac{\int y\, ds}{\int ds}, \text{ by Art. 128 (8).}$$

Therefore
$$2\pi\bar{y}\cdot s = 2\pi\int y\, ds. \tag{1}$$

But by Art. 124 (1), the second member of (1) is the area of the surface generated by the revolution of the curve whose length is s about the X-axis, and the first member is the circumference described by the centre of mass, multiplied by the length of the curve s.

Hence, if a plane curve revolve about an axis in its own plane external to itself, the area of the surface generated is equal to the length of the revolving curve, multiplied by the circumference described by its centre of mass.

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* Sometimes called Theorems of Pappus, as they were first stated by Pappus.

II. A plane area revolves about the X-axis. The ordinate of the centre of mass of the plane surface is

$$\bar{y} = \frac{\int\int y\,dx\,dy}{\int\int dx\,dy}, \quad \text{by Art. 129 (2).}$$

Therefore $$2\pi\bar{y}\int\int dx\,dy = 2\pi\int\int y\,dx\,dy = \pi\int y^2dx. \qquad (2)$$

But by Art. 124 (5), the last member of (2) is the volume generated by the revolution of the area; and in the first member, $\int\int dx\,dy$ is the revolving area. Hence, if a plane area revolve about an axis external to itself, the volume generated is equal to the area of the revolving figure, multiplied by the circumference described by its centre of mass.

If the curve or area revolve through any angle θ instead of making an entire revolution, θ must be substituted for 2π in equations (1) and (2).

PROBLEMS.

1. Find the surface and volume of the ring generated by revolving a circle whose radius $= r$, about an external axis distant b from the centre of the circle. *Ans.* $S = 4\pi^2ab$, $V = 2\pi^2a^2b$.

2. Find the volume generated by an ellipse revolved about an axis distant 10 from the centre; the semi-axes being 10 and 5. *Ans.* 9869.6+.

3. Find the surface and volume generated by revolving a cycloid about its base. *Ans.* $S = \frac{64}{3}\pi a^2$, $V = 5\pi^2a^3$.

CHAPTER XXII.

DIFFERENTIAL EQUATIONS.

ART. 134. DEFINITIONS.

A *differential equation* between two variables x and y is an equation containing one or both of the variables x and y and one or more derivatives, such as $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$, etc.

The *order* of a differential equation is that of the highest derivative which it contains.

The *degree* of a differential equation is that of the highest power to which the highest derivative which it contains is raised.

The solution of a differential equation consists in finding a relation between x and y and constants, from which the given equation may be derived by differentiation; this relation is called the primitive. The solution requires one or more integrations, and each integration introduces an arbitrary constant; hence the solution of a differential equation of the nth order will give an equation containing n arbitrary constants.

The same primitive may have several differential equations of the same order.

For example, given the equation

$$ay + bx + c = 0. \tag{1}$$

By differentiating,

$$a\frac{dy}{dx} + b = 0. \tag{2}$$

Eliminating a between (1) and (2),

$$bx\frac{dy}{dx} + c\frac{dy}{dx} - by = 0. \tag{3}$$

Eliminating b between (1) and (2),

$$ay + c - ax\frac{dy}{dx} = 0. \tag{4}$$

In this example, equation (1) is called the complete primitive, and equations (2), (3) and (4) are differential equations showing the same relation between the variables.

ART. 135. DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND DEGREE.

The general form of the differential equation of the first order and degree is

$$Mdx + Ndy = 0, \tag{1}$$

in which M and N are functions of x and y. This equation may be put in the form

$$M + N\frac{dy}{dx} = 0.$$

The most obvious method of solving a differential equation of the first order and degree is by means of the separation of the variables, whenever practicable. The variables are separated when the coefficient of dx contains the variable x only, and the coefficient of dy contains the variable y only; that is, when the equation can be reduced to the form

$$Xdx + Ydy = 0,$$

in which X is a function of x only, and Y is a function of y only.

Let the form be

$$XYdx + X'Y'dy = 0, \tag{2}$$

in which X and X' are functions of x only, and Y and Y' are functions of y only.

Dividing by $X'Y$,

$$\frac{Xdx}{X'} + \frac{Y'}{Y}dy = 0, \tag{3}$$

in which equation the variables are separated.

For example, given

$$(1 - x)^2 y\,dx - (1 + y)\,x^2 dy = 0.$$

Dividing by x^2y,

$$\frac{(1 - x)^2}{x^2}dx - \frac{1 + y}{y}dy = 0.$$

Hence, $$\frac{dx}{x^2} - \frac{2\,dx}{x} + dx - \frac{dy}{y} - dy = 0.$$

Integrating, $$-\frac{1}{x} - 2\log x + x - \log y - y = C.$$

PROBLEMS.

1. $(1-y)\,dx + (1+x)\,dy = 0.$ *Ans.* $\log(1+x) - \log(1-y) = C.$

2. $(1+y^2)\,dx - x^{\frac{1}{2}}dy = 0.$ *Ans.* $2x^{\frac{1}{2}} - \text{arc}\tan y = C.$

3. $(1+x)y\,dx + (1-y)x\,dy = 0.$ *Ans.* $\log(xy) + x - y = C.$

4. $dy + y\tan x\,dx = 0.$ *Ans.* $\log y - \log\cos x = C.$

5. $\dfrac{dy}{dx} = \dfrac{1+y^2}{(1+x^2)xy}.$ *Ans.* $(1+x^2)(1+y^2) = Cx^2.$

6. $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}.$ *Ans.* $y = \dfrac{x+C}{1-Cx}.$

7. $\sin x\cos y\,dx - \cos x\sin y\,dy = 0.$ *Ans.* $\cos y = C\cos x.$

8. Helmholtz's equation for the strength of an electric current C at the time t is $C = \dfrac{E}{R} - \dfrac{L}{R}\dfrac{dC}{dt}$, in which E, R and L are given constants. Find the value of C, determining the constant of integration by the condition that its initial value shall be zero.

9. The equation showing the strength of current i for the time t after source of E. M. F. is removed, is $RC = -L\dfrac{dC}{dt}$ (R and L being constants). Find the value of C.

Ans. $C = Ie^{-\frac{Rt}{L}}$, in which $I =$ current when $t = 0$.

10. The differential equation of the current of discharge from a condenser of capacity C in a circuit of resistance R is $\dfrac{di}{i} = \dfrac{dt}{CR}$.

Find i, if the initial current is I_0. *Ans.* $i = I_0e^{\frac{t}{CR}}$.

ART. 136. HOMOGENEOUS DIFFERENTIAL EQUATIONS.

The differential equation $Mdx + Ndy = 0$ is said to be homogeneous when M and N are homogeneous functions of x and y of the same degree.

If the equation is written in the form

$$\frac{dy}{dx} = -\frac{M}{N},$$

the second member is seen to be a function of $\frac{y}{x}$.

If, now, v be substituted for $\frac{y}{x}$,

$$y = vx, \text{ and } \frac{dy}{dx} = x\frac{dv}{dx} + v,$$

and the equation becomes

$$x\frac{dv}{dx} + v = f(v),$$

in which the variables can be separated, giving

$$\frac{dx}{x} = \frac{dv}{f(v) - v}.$$

For example, given the homogeneous equation,

$$y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}.$$

Substituting $y = vx$,

$$v^2x^2 + \frac{x^2(x\,dv + v\,dx)}{dx} = \frac{vx^2(x\,dv + v\,dx)}{dx};$$

whence $$\frac{dv}{v} + \frac{dx}{x} = dv.$$

Integrating and substituting $v = \frac{y}{x}$,

$$\log\frac{y}{x} + \log x = \frac{y}{x} + c.$$

Therefore $$\log y - c = \frac{y}{x},$$

or $$c_1 e^{\frac{y}{x}} = y \cdot (\log c_1 = c).$$

PROBLEMS.

1. $(x - 2y)\,dx + y\,dy = 0.$ *Ans.* $\log(x - y) - \frac{x}{y - x} = c.$

2. $(2\sqrt{xy} - x)\,dy + y\,dx = 0.$ *Ans.* $y = ce^{-\sqrt{\frac{x}{y}}}.$

3. $x^2dy - y^2dx - xy\,dx = 0.$ *Ans.* $\log x + \frac{x}{y} = c.$

4. $x\,dy - y\,dx = dx\sqrt{x^2 - y^2}.$ *Ans.* $\log x = \text{arc}\sin\frac{y}{x} + c.$

5. $(8y + 10x)dx + (5y + 7x)dy = 0.$ *Ans.* $(y + x)^2(y + 2x)^3 = c.$

6. $(x^2 + y^2)dx - 2xy\,dy = 0.$ *Ans.* $x^2 - y^2 = cx.$

7. Find the curve in which the subtangent is equal to the sum of the abscissa and ordinate at any point of the curve.

From Art. 70 (3), it is seen that the differential equation is

$$-y\frac{dx}{dy} = x + y;$$

whence $y^2 + 2xy = c$, a hyperbola.

ART. 137. THE FORM $(ax + by + c)dx + (a'x + b'y + c')dy = 0.$

The equation $Mdx + Ndy = 0$ can always be solved when M and N are functions of x and y of the first degree, or having the form

$$(ax + by + c)dx + (a'x + b'y + c')dy = 0. \qquad (1)$$

Assuming $x = x' + h$ and $y = y' + k$, and substituting in (1),

$$(ax' + by' + ah + bk + c)dx' + (a'x' + b'y' + a'h + b'k + c')dy' = 0. \qquad (2)$$

In order that (2) may be homogeneous,

$$ah + bk + c = 0, \text{ and } a'h + b'k + c' = 0,$$

giving $$h = \frac{cb' - c'b}{a'b - ab'}, \text{ and } k = \frac{ac' - a'c}{a'b - ab'}.$$

Equation (2) now becomes

$$(ax' + by')dx' + (a'x' + b'y')dy' = 0,$$

a homogeneous equation, and the variables can be separated as in the preceding article.

This method evidently fails when $a'b = ab'$; that is, when $\frac{a'}{a} = \frac{b'}{b}$. In this case put $\frac{a'}{a} = \frac{b'}{b} = m$, $a' = ma$, $b' = mb$.

Equation (1) now takes the form

$$(ax + by + c)dx + [m(ax + by) + c']dy = 0. \qquad (3)$$

Assuming $\quad ax + by = z,\ dy = \dfrac{dz - a\,dx}{b},$

and substituting in (3),

$$(z + c)\,dx + (mz + c')\,\frac{dz - a\,dx}{b} = 0;$$

whence $\quad [b(z + c) - a(mz + c')]\,dx + (mz + c')\,dz = 0,$

and $$dx + \frac{(mz + c')\,dz}{b(z + c) - a(mz + c')} = 0,$$

and the variables are separated.

PROBLEMS.

1. $$(1 + x + y)\,dx + (1 + 2x + 3y)\,dy = 0.$$

Assuming $\quad x = x' + h$, and $y = y' + k$,

$$(1 + x' + h + y' + k)\,dx' + (1 + 2x' + 2h + 3y' + 3k)\,dy' = 0,$$

in which $\quad h + k + 1 = 0$, and $2h + 3k + 1 = 0$,

giving $\quad h = -2$, and $k = 1$.

The equation now becomes

$$(x' + y')\,dx' + (2x' + 3y')\,dy' = 0,$$

which is homogeneous and can be solved by Art. 136.

2. $\dfrac{dy}{dx} = ax + by + c.$ *Ans.* $abx + b^2y + a + bc = Ce^{bx}.$

3. $(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0.$

Ans. $x + 2y + \log(2x + y - 1) = C.$

4. $(2x - y + 1)\,dx + (2y - x - 1)\,dy = 0.$

Ans. $x^2 - xy + y^2 + x - y = C.$

ART. 138. THE LINEAR EQUATION OF THE FIRST ORDER.

The equation of the form

$$\frac{dy}{dx} + Py = Q, \tag{1}$$

in which P and Q are functions of x only, is called a linear equation because it is of the first degree with respect to y and its derivative.

This linear equation admits of a general solution. As the second member is a function of x only, an integrating factor of the first member will be an integrating factor of the equation if it is a function of x only. To find such a factor, put $y = Xz$, in which X is an arbitrary function of x, and z is a new variable. Then $dy = Xdz + zdX$, which reduces (1) to

$$Xdz + zdX + PXzdx = Qdx. \qquad (2)$$

Assume $$zdX = Qdx, \qquad (3)$$

then (2) becomes $$Xdz + PXzdx = 0;$$

whence $$\frac{dz}{z} = -Pdx; \text{ hence } \log z = -\int Pdx, \text{ and } z = e^{-\int Pdx}.$$

Substituting this value of z in (3),

$$e^{-\int Pdx}dX = Qdx, \text{ or } dX = e^{\int Pdx}Qdx.$$

Therefore $X = \int e^{\int Pdx}Qdx + c$, or $y = Xz = e^{-\int Pdx}\left(\int e^{\int Pdx}Qdx + c\right)$.

PROBLEMS.

1. $dy - \dfrac{yx\,dx}{1+x^2} = \dfrac{a}{1+x^2}dx.$

This linear equation for y, put in the general form, is

$$\frac{dy}{dx} - \frac{yx}{1+x^2} = \frac{a}{1+x^2},$$

in which $\int Pdx = -\int \dfrac{x\,dx}{1+x^2} = \log \dfrac{1}{\sqrt{1+x^2}}.$

Hence the integrating factor is

$$e^{\int Pdx} = e^{\log(1+x^2)^{-\frac{1}{2}}} = (1+x^2)^{-\frac{1}{2}}.$$

$$\int e^{\int Pdx}Qdx = \int \frac{a(1+x^2)^{-\frac{1}{2}}}{1+x^2}dx = \frac{ax}{(1+x^2)^{\frac{1}{2}}} + c.$$

Therefore $y = (1+x^2)^{\frac{1}{2}}\left(\dfrac{ax}{(1+x^2)^{\frac{1}{2}}} + c\right) = ax + c(1+x^2)^{\frac{1}{2}}.$

2. $x^2\dfrac{dy}{dx}+(1-2x)y=x^2.$ *Ans.* $y=x^2(1+ce^{\frac{1}{x}}).$

3. $x\dfrac{dy}{dx}-ay=x+1.$ *Ans.* $y=cx^a-\dfrac{x}{a-1}-\dfrac{1}{a}.$

4. $\sqrt{a^2+x^2}-\sqrt{a^2+x^2}\dfrac{dy}{dx}=x+y.$

Ans. $y=\dfrac{a^2\log(x+\sqrt{a^2+x^2})+c}{x+\sqrt{a^2+x^2}}.$

5. $\dfrac{dy}{dx}\cos x+y\sin x=1.$ *Ans.* $y=\sin x+c\cos x.$

6. The differential equation of electromotive forces in a circuit of resistance R and induction L, when the impressed E. M. F. is a sinusoid given by the equation $e=E\sin pt$, is $E\sin pt=Ri+L\dfrac{di}{dt}$. Solve the equation for i, i and t being the only variables.

Ans. $i=ce^{-\frac{Rt}{L}}+\dfrac{E}{\sqrt{R^2+p^2L^2}}\sin(pt-\phi)$, in which $\phi=\tan^{-1}\dfrac{Lp}{R}.$

ART. 139. EXTENSION OF THE LINEAR EQUATION.

The more general form, $$\frac{dy}{dx}+Py=Qy^n, \tag{1}$$

in which P and Q are functions of x only, is readily put in the form of the linear equation of the preceding article.

Dividing (1) by y^n, $$\frac{1}{y^n}\frac{dy}{dx}+\frac{P}{y^{n-1}}=Q. \tag{2}$$

Assume

$z=y^{-n+1}$; whence $y^n=z^{-\frac{n}{n-1}}$, $y^{n-1}=z^{-1}$, and $dz=-(n-1)y^{-n}dy.$

Substituting in (2) and reducing,

$$\frac{dz}{dx} - (n-1)Pz = -(n-1)Q,$$

which is the form of the linear equation of the preceding article.

PROBLEMS.

1. $dy + y\,dx = xy^3dx.$

Dividing by y^3dx,

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2} = x.$$

Assume $$z = y^{-2}.$$

Substituting and reducing,

$$\frac{dz}{dx} - 2z = -2x.$$

By the preceding article,

$$\int P dx = -2x, \text{ and}$$

$$z = -2e^{2x}\int e^{-2x}x\,dx = e^{2x}(xe^{-2x} + \tfrac{1}{2}e^{-2x} + C).$$

Therefore $$1 = y^2(Ce^{2x} + \tfrac{1}{2} + x).$$

2. $\dfrac{dy}{dx} = x^3y^3 - xy.$ *Ans.* $\dfrac{1}{y^2} = x^2 + 1 + Ce^{x^2}.$

3. $3y^2\dfrac{dy}{dx} - ay^3 = x + 1.$ *Ans.* $y^3 = Ce^{ax} - \dfrac{x+1}{a} - \dfrac{1}{a^2}.$

4. $\dfrac{dy}{dx}(x^2y^3 + xy) = 1.$ *Ans.* $x = \dfrac{e^{\frac{y^2}{2}}}{(2-y^2)e^{\frac{y^2}{2}} + C}.$

ART. 140. EXACT DIFFERENTIAL EQUATIONS.

The equation $$Mdx + Ndy = 0 \tag{1}$$

is an exact differential equation, when

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{2}$$

When the condition in (2) is fulfilled, the integral may be obtained by finding $\int Mdx$, regarding y as constant and adding an arbitrary function of y.

The undetermined function of y may be found by the condition that the differential of the result just obtained regarding x as constant must equal Ndy; that is,

$$\frac{\partial}{\partial y}\int Mdx + \frac{df(y)}{dy} = N,$$

from which $f(y)$ may be obtained. (See Art. 49.)

PROBLEMS.

1. $\dfrac{dx}{y} + \left(2y - \dfrac{x}{y^2}\right)dy = 0.$

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2}, \text{ and } \frac{\partial N}{\partial x} = -\frac{1}{y^2};$$

condition (2) is fulfilled, and the equation is exact. $\int Mdx$, treating y as constant and adding $f(y)$, is $\dfrac{x}{y} + f(y)$.

Now $$\frac{\partial}{\partial y}\left(\frac{x}{y}\right) + \frac{df(y)}{dy} = 2y - \frac{x}{y^2};$$

whence $$-\frac{x}{y^2} + \frac{df(y)}{dy} = 2y - \frac{x}{y^2},$$

and $$\frac{df(y)}{dy} = 2y;$$

hence $$f(y) = y^2.$$

Therefore $$\frac{x}{y} + y^2 = C.$$

2. $(6xy - y^2)\,dx + (3x^2 - 2xy)\,dy = 0.$ *Ans.* $3x^2y - y^2x = 0.$

3. $x(x + 2y)\,dx + (x^2 - y^2)\,dy = 0.$ *Ans.* $x^3 + 3x^2y - y^3 = 0.$

4. $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0.$
Ans. $x^3 - 6x^2y - 6xy^2 + y^3 = 0.$

5. $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0.$ *Ans.* $x^2 + y^2 - 2\arctan\dfrac{y}{x} = C.$

6. $e^x(x^2 + y^2 + 2x)\,dx + 2ye^x dy = 0.$ *Ans.* $e^x(x^2 + y^2) = C.$

ART. 141. FACTORS NECESSARY TO MAKE DIFFERENTIAL EQUATIONS EXACT.

When $Mdx + Ndy$ is not an exact differential, it may often be transformed into an exact differential by the introduction of a factor containing x or y or both. This factor, which converts a given differential equation into an exact differential equation, is called an *integrating factor.*

I. *When $Mdx + Ndy$ is homogeneous.*

$$Mdx + Ndy = \tfrac{1}{2}\left[(Mx + Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$$

$$= \tfrac{1}{2}\left[(Mx + Ny)\, d\log(xy) + (Mx - Ny)\, d\log\frac{x}{y}\right]. \quad (1)$$

Hence $$\frac{Mdx + Ndy}{Mx + Ny} = \tfrac{1}{2}\, d\log(xy) + \tfrac{1}{2}\,\frac{Mx - Ny}{Mx + Ny}\, d\log\frac{x}{y}$$

$$= \tfrac{1}{2}\, d\,[\log v + \log y^2] + \tfrac{1}{2}\,\frac{Mx - Ny}{Mx + Ny}\,\frac{dv}{v}, \text{ when } v = \frac{x}{y}. \quad (2)$$

When M and N are homogeneous,

$$\frac{Mx - Ny}{Mx + Ny} = f(v),$$

and the second member of (2) is an exact differential.

Therefore $\dfrac{1}{Mx + Ny}$ is an integrating factor.

This method fails when $Mx + Ny = 0$, but in this case $Mx = -Ny$. Dividing the first term of $Mdx + Ndy = 0$ by Mx, and the second term by its equal, $-Ny$,

$$\frac{dx}{x} - \frac{dy}{y} = 0.$$

Therefore $$y = Cx.$$

For example, given $(xy + y^2)\, dx - (x^2 - xy)\, dy = 0.$

Here $$\frac{1}{Mx + Ny} = \frac{1}{2\,xy^2}.$$

Multiplying by this factor,

$$\frac{1}{2}\left(\frac{1}{y}+\frac{1}{x}\right)dx-\frac{1}{2}\left(\frac{x}{y^2}-\frac{1}{y}\right)dy=0.$$

The condition of integrability for an exact differential is now fulfilled.

Therefore $$\frac{x}{y}+\log(xy)=C.$$

II. *The form,* $f_1(xy)\,y\,dx+f_2(xy)\,x\,dy=0.$

By a method similar to that of I., it may be shown that $\dfrac{1}{Mx-Ny}$ is an integrating factor. This fails when $Mx-Ny=0$, but it can then be shown, as in the corresponding case of I., that the solution now is $xy=C$.

But another method of solution by the separation of the variables may be used. For example, given

$$(x^2y^2+xy)\,y\,dx+(x^2y^2-1)\,x\,dy=0. \tag{3}$$

Assume $$xy=v,$$

then $$(v^2+v)\frac{v}{x}dx+(v^2-1)x\left(\frac{dv}{x}-\frac{v\,dx}{x^2}\right)=0;$$

whence $$\frac{dx}{x}=-\left(\frac{v-1}{v}\right)dv, \tag{4}$$

and the variables are separated.

Integrating (4), and substituting $v=xy$,

$$y=ce^{xy}.$$

III. *To determine the factor necessary to render $Mdx+Ndy$ exact, when that factor is a function of one variable only.*

Assume X, a function of x only, to be the required factor; then $XMdx+XNdy$ is an exact differential.

Hence $$\frac{\partial}{\partial y}(XM)=\frac{\partial}{\partial x}(XN);$$

and since $$\frac{\partial X}{\partial y}=0,$$

$$X\frac{\partial M}{\partial y}=X\frac{\partial N}{\partial x}+N\frac{dX}{dx};$$

therefore
$$\frac{dX}{X}=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx. \tag{5}$$

The first member of (5) does not contain y; hence the second member must be independent of y also, or

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx=f(x).$$

Integrating (5), $\log X=\int f(x)\,dx,$

therefore $X=e^{\int f(x)dx}.$

Similarly, if Y, a function of y only, is the original factor,

$$YMdx+YNdy$$

is an exact differential, and

$$Y=e^{\int\phi(y)dy}.$$

For example, given $(x^2+y^2+2x)\,dx+2y\,dy=0.$

Assume X to be the integrating factor.

$$\frac{\partial M}{\partial y}=2y,\ \frac{\partial N}{\partial x}=0;$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx=dx.$$

$$\log X=\int dx=x, \text{ and } X=e^x.$$

Multiplying the given equation by e^x,

$$e^x(x^2+y^2+2x)\,dx+2e^xy\,dy=0.$$

The condition of integrability for an exact differential is now fulfilled, and integrating as in Art. 140,

$$e^xx^2+e^xy^2=c.$$

PROBLEMS.

1. $x(x^2+3y^2)dx+y(y^2+3x^2)dy=0.$ *Ans.* $x^4+6x^2y^2+y^4=c.$

2. $y\,dy+x\,dx+\dfrac{x\,dy-y\,dx}{x^2+y^2}.$ *Ans.* $\dfrac{x^2+y^2}{2}+\arctan\dfrac{y}{x}=c.$

3. $(x^2+y^2)dx-2xy\,dy=0.$ *Ans.* $x^2-y^2=cx.$

4. $(x^2+2xy-y^2)dx=x^2-2xy-y^2dy.$ *Ans.* $x^2+y^2=c(x+y).$

5. $(x+y)^2\dfrac{dy}{dx}=a^2.$ *Ans.* $y-a\arctan\dfrac{x+y}{a}=c.$

6. $x^3dx+(3x^2y+2y^3)dy=0.$ *Ans.* $x^2+2y^2=c\sqrt{x^2+y^2}.$

7. $(1+xy)y\,dx+(1-xy)x\,dy=0.$ *Ans.* $x=cye^{\frac{1}{xy}}.$

8. $(y+y\sqrt{xy})dx+(x+x\sqrt{xy})dy=0.$ *Ans.* $xy=c.$

9. $(3x^2-y^2)\dfrac{dy}{dx}=2xy.$ *Ans.* $x^2-y^2=cy^3.$

10. $2xy\,dy=(x^2+y^2)dx.$ *Ans.* $x^2-y^2=cx.$

ART. 142. FIRST ORDER AND DEGREE WITH THREE VARIABLES.

The general form of the differential equation of the first order and degree between three variables is

$$Pdx+Qdy+Rdz=0, \tag{1}$$

in which P, Q and R are functions of x, y and z.

Such an equation sometimes admits of solution by the separation of the variables.

To obtain the condition of integrability, represent the function by

$$v=f(x,y,z)=c; \tag{2}$$

whence
$$\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz=0. \tag{3}$$

Comparing (1) and (3), it is seen that

P, Q and R are proportional to $\dfrac{\partial v}{\partial x}$, $\dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial z}$.

To obtain the required relation between P, Q and R, assume the factor u such that

$$uP = \frac{\partial v}{\partial x}, \tag{4}$$

$$uQ = \frac{\partial v}{\partial y}, \tag{5}$$

and

$$uR = \frac{\partial v}{\partial z}. \tag{6}$$

From (4) and (5),

$$\frac{\partial}{\partial y}(uP) = \frac{\partial^2 v}{\partial y\, \partial x} = \frac{\partial}{\partial x}(uQ);$$

hence

$$u\frac{\partial P}{\partial y} + \frac{P\partial u}{\partial y} = u\frac{\partial Q}{\partial x} + Q\frac{\partial u}{\partial x},$$

or

$$u\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\partial u}{\partial x} - P\frac{\partial u}{\partial y}. \tag{7}$$

Similarly,

$$u\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = R\frac{\partial u}{\partial y} - Q\frac{\partial u}{\partial z}, \tag{8}$$

and

$$u\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = P\frac{\partial u}{\partial z} - R\frac{\partial u}{\partial x}. \tag{9}$$

Multiplying equations (7), (8) and (9) by R, P and Q, respectively, and adding,

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0. \tag{10}$$

Equation (10) is the required condition of integrability. When this condition is fulfilled, equation (1) may be integrated by regarding one of the variables, x, y or z, as constant, and omitting the corresponding term, Pdx, Qdy or Rdz.

Thus omitting Rdz, integrating $Pdx + Qdy = 0$, regarding z as constant and introducing $f(z)$ as the constant of integration, the integral is obtained so far as it depends upon x and y. Finally, by comparing the total differential of this result with equation (1), $df(z)$ is found in terms of z and dz, and then by integration the value of $f(z)$.

When certain terms of the equation form an exact differential, the remaining terms must also be exact. It follows that if one of the vari-

ables, say z, can be completely separated from the other two, so that in equation (1) R becomes a function of z only, and P and Q functions of x and y only, the terms $Pdx + Qdy$ must be thus rendered exact if the equation is integrable.

For example, given $zy\,dx - zx\,dy - y^2dz = 0$. Dividing by y^2z, which separates z from x and y, and puts it in the exact form,

$$\frac{y\,dx - x\,dy}{y^2} - \frac{dz}{z} = 0,$$

of which the integral is $x = y \log cz$.

PROBLEMS.

1. $(x - 3y - z)\,dx + (2y - 3x)\,dy + (z - x)\,dz = 0$.
   Ans. $x^2 + 2y^2 - 6xy - 2xz + z^2 = C$.

2. $yz^2dx - z^2dy - e^x dz = 0$. *Ans.* $yz = e^x(1 + cz)$.

3. $yz\,dx + zx\,dy + yx\,dz = 0$. *Ans.* $xyz = C$.

4. $(y\,dx + x\,dy)(a + z) = xy\,dz$. *Ans.* $xy = c\,(a + z)$.

5. The increase in energy of a magnetic field caused by the independent increments di_1 and di_2 of current in two coils of self-induction L_1 and L_2 and mutual induction μ is $dW = L_1 i_1\,di_1 + L_2 i_2\,di_2 + \mu i_1\,di_2 + \mu i_2\,di_1$. Find the energy of the field.
   Ans. $W = \frac{L_1 i_1^2}{2} + \frac{L_2 i_2^2}{2} + \mu i_1 i_2$.

ART. 143. FIRST ORDER AND SECOND DEGREE.

The general form of a differential equation of the first order and second degree is

$$\frac{dy^2}{dx^2} + M\frac{dy}{dx} + N = 0, \tag{1}$$

in which M and N are functions of x and y. The direct differential obtained from the primitive contains only the first power of $\frac{dy}{dx}$, and hence cannot be identical with (1). But if the primitive is supposed to contain the first and second powers of a constant c, and is solved with reference to c, there will result two values of c, from each of which c will disappear on differentiation; and each of these resulting differ-

ential equations will contain only the first power of $\frac{dy}{dx}$, each being a factor of (1). Hence the product of the two equations will give (1).

Therefore the given equation is solved as a quadratic in $\frac{dy}{dx}$, all the terms are transposed to the first member, and it is resolved into two factors of the first order and degree. Each of these factors is then placed separately equal to zero and integrated, using the same arbitrary constant in each. When all the terms in each of these results are transposed to the first member, the product of these first members placed equal to zero will be the complete primitive.

For example, given $$y\frac{dy^2}{dx^2} + 2x\frac{dy}{dx} - y = 0. \quad (1)$$

Solving for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = -\frac{x}{y} + \frac{\sqrt{x^2+y^2}}{y}, \text{ and } \frac{dy}{dx} = -\frac{x}{y} - \frac{\sqrt{x^2+y^2}}{y};$$

whence $$dx = +\frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}, \quad \text{and } dx = -\frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}. \quad (2)$$

Integrating (2),

$$x = +\sqrt{x^2+y^2} + c, \text{ and } \quad x = -\sqrt{x^2+y^2} + c.$$

Therefore $$(x - c - \sqrt{x^2+y^2})(x - c + \sqrt{x^2+y^2}) = 0,$$

or $$y^2 = c^2 - 2cx.$$

PROBLEMS.

1. $\frac{dy^2}{dx^2} = ax.$ *Ans.* $(y-c)^2 = \frac{4}{9}ax^3.$

2. $\frac{dy^2}{dx^2} - 5\frac{dy}{dx} + 6 = 0.$ *Ans.* $(y - 2x + c)(y - 3x + c) = 0.$

3. $x^2\frac{dy^2}{dx^2} + 3xy\frac{dy}{dx} + 2y^2 = 0.$ *Ans.* $(xy + c)(x^2y + c) = 0.$

4. $(x^2+1)\frac{dy^2}{dx^2} = 1.$ *Ans.* $c^2e^{2y} - 2cxe^y = 1.$

5. $\frac{dy}{dx}\left(\frac{dy}{dx} + y\right) = x(x+y).$ *Ans.* $(x^2 - 2y + c)[e^x(x + y - 1) + c].$

ART. 144. DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

I. *Equations involving* x *and* $\frac{d^2y}{dx^2}$ *only.*

The equation, if possible, is put in the form

$$\frac{d^2y}{dx^2} = X, \tag{1}$$

in which X is a function of x only.

Integrating (1), $$\frac{dy}{dx} = X_1 + C_1. \tag{2}$$

Integrating (2), $$y = X_2 + C_1x + C_2. \tag{3}$$

In (2) and (3), X_1 and X_2 are functions of x only, and C_1 and C_2 are arbitrary constants.

For example, given $$\frac{d^2y}{dx^2} = ax^4.$$

Then $$\frac{d^2y}{dx} = ax^4dx,$$

$$\frac{dy}{dx} = \frac{ax^5}{5} + C_1,$$

$$dy = \frac{ax^5dx}{5} + C_1dx,$$

$$y = \frac{ax^6}{30} + C_1x + C_2.$$

II. *Equations involving* y *and* $\frac{d^2y}{dx^2}$ *only.*

The equation, if possible, is put in the form

$$\frac{d^2y}{dx^2} = Y, \tag{4}$$

in which Y is a function of y only.

Multiplying (4) by $2\frac{dy}{dx}$,

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2Y\frac{dy}{dx};$$

whence $$2\frac{dy}{dx}d\left(\frac{dy}{dx}\right) = 2\,Ydy. \tag{5}$$

Integrating (5),

$$\left(\frac{dy}{dx}\right)^2 = 2\int Ydy + C_1.$$

Separating the variables and integrating,

$$x = \int \frac{dy}{\sqrt{2\int Ydy + C_1}} + C_2.$$

For example, given $\frac{d^2y}{dx^2} = a^2y.$

Then $$2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2\,a^2y\frac{dy}{dx},$$

$$2\frac{dy}{dx}\,d\left(\frac{dy}{dx}\right) = 2\,a^2y\,dy,$$

$$\left(\frac{dy}{dx}\right)^2 = a^2y^2 + C_1,$$

$$dx = \frac{dy}{\sqrt{a^2y^2 + C_1}},$$

$$ax = \log\left(y + \sqrt{y^2 + \frac{C_1}{a^2}} + C_2\right).$$

III. *Equations not involving y directly.*

The equation will be of the form $F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0.$

Assume $\frac{dy}{dx} = z$, then $\frac{d^2y}{dx^2} = \frac{dz}{dx}.$

Making these substitutions, an equation of the first order between z and x is obtained.

For example, given $\frac{d^2y}{dx^2} + \frac{2x}{a^2}\sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3} = 0.$

Assume $$\frac{dy}{dx} = z,$$

then $$\frac{dz}{\sqrt{(1 + z^2)^3}} = -\frac{2x}{a^2}dx.$$

Integrating,

$$\frac{z}{\sqrt{1 + z^2}} = -\frac{x^2}{a^2} + C_1 = \frac{c^2 - x^2}{a^2} \text{ when } C_1 = \frac{c^2}{a^2};$$

whence $$z = \frac{dy}{dx} = \frac{c^2 - x^2}{\sqrt{a^4 - (c^2 - x^2)^2}}.$$

Therefore $$y = \int \frac{(c^2 - x^2)\,dx}{\sqrt{a^4 - (c^2 - x^2)^2}}.$$

IV. *Equations not involving x directly.*

Assume $$\frac{dy}{dx} = z,$$

then $$\frac{d^2y}{dx^2} = \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = z\frac{dz}{dy}.$$

Making these substitutions, the independent variable is changed from x to y, and an equation of the first order between z and y is obtained.

For example, given $$\frac{d^2y}{dx^2} - a\left(\frac{dy}{dx}\right)^2 = y. \tag{6}$$

Substituting $\frac{dy}{dx} = z$, and $\frac{d^2y}{dx^2} = z\frac{dz}{dy}$,

$$z\frac{dz}{dy} - az^2 = y. \tag{7}$$

Assume $z^2 = 2v$, then $z\,dz = dv$, and substituting in (7),

$$dv - 2av\,dy = y\,dy. \tag{8}$$

Equation (8) is a linear equation of the first order and degree and is therefore integrable.

PROBLEMS.

1. $\frac{d^2y}{dx^2} = 6x.$ *Ans.* $y = x^3 + C_1x + C_2.$

2. $\frac{d^2y}{dx^2} = e^{ny}.$ *Ans.* $x = \frac{C_1}{\sqrt{2n}}\log\frac{\sqrt{C_1^2e^{ny}+1}-1}{\sqrt{C_1^2e^{ny}+1}+1} + C_2.$

3. $\frac{d^2y}{dx^2} = \frac{1}{\sqrt{ay}}.$ *Ans.* $x = \sqrt[4]{a}\left[\frac{2}{3}\left(y^{\frac{1}{2}} + \frac{4C_1}{a^{\frac{1}{2}}}\right) - \frac{8C_1}{a^{\frac{1}{2}}}\left(y^{\frac{1}{2}} + \frac{4C_1}{a^{\frac{1}{2}}}\right)\right] + C_2.$

4. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0.$ *Ans.* $y = C_1\log x + C_2.$

5. $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=2.$ *Ans.* $y=(\text{arc}\sin x)^2+C_1\,\text{arc}\sin x+C_2.$

6. $a^2\left(\dfrac{d^2y}{dx^2}\right)^2=1+\left(\dfrac{dy}{dx}\right)^2.$ *Ans.* $\dfrac{2y}{a}=C_1e^{\frac{x}{a}}+\dfrac{e^{-\frac{x}{a}}}{C_1}+C_2.$

7. $y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=1.$ *Ans.* $y^2=x^2+C_1x+C_2.$

8. $y(1-\log y)\dfrac{d^2y}{dx^2}+(1+\log y)\left(\dfrac{dy}{dx}\right)^2=0.$

Ans. $\log y-1=\dfrac{1}{C_1x+C_2}.$

9. $y\dfrac{d^2y}{dx^2}-\left(\dfrac{dy}{dx}\right)^2=y^2\log y.$ *Ans.* $\log y=C_1e^x+C_2e^{-x}.$

10. The equation of motion of a particle ascending in the air against the action of gravity is

$$\frac{d^2x}{dt^2}=-g-gk^2\left(\frac{dx}{dt}\right)^2.$$

Find the equation for the space described by the particle in terms of the time; determining the constants of integration by making $\dfrac{dx}{dt}=v$ when $t=0$, and $x=0$ when $t=0$.

Ans. $x=\dfrac{1}{gk^2}\log(vk\sin kgt+\cos kgt).$

APPENDIX.

TABLE OF INTEGRALS.

This table contains the principal integrals given in this book, together with a few additional ones. The arrangement only partially follows the order in which the forms occur in the book, as convenience of reference is the first consideration.

I.

ELEMENTARY FORMS.

1. $\int (du + dv - dw) = u + v - w.$

2. $\int a\,dx = ax.$

3. $\int a df(x) = a f(x).$

4. $\int a \frac{dx}{x} = a \log x.$

5. $\int ax^n dx = \frac{ax^{n+1}}{n+1}.$

EXPONENTIAL FORMS.

6. $\int a^x \log a\,dx = a^x.$

7. $\int e^x dx = e^x.$

TRIGONOMETRIC FORMS.

8. $\int \cos x\,dx = \sin x.$

9. $\int \sec^2 x\,dx = \tan x.$

10. $\int \sin x\,dx = -\cos x.$

11. $\int \operatorname{cosec}^2 x\,dx = -\cot x.$

12. $\int \sec x \tan x \, dx = \sec x.$

13. $\int \operatorname{cosec} x \cot x \, dx = - \operatorname{cosec} x.$

14. $\int \sin x \, dx = \operatorname{vers} x.$

15. $\int \cos x \, dx = - \operatorname{covers} x.$

INVERSE TRIGONOMETRIC FUNCTIONS.

16. $\int \frac{dx}{\sqrt{1-x^2}} = \operatorname{arc} \sin x.$

17. $\int - \frac{dx}{\sqrt{1-x^2}} = \operatorname{arc} \cos x.$

18. $\int \frac{dx}{1+x^2} = \operatorname{arc} \tan x.$

19. $\int - \frac{dx}{1+x^2} = \operatorname{arc} \cot x.$

20. $\int \frac{dx}{x\sqrt{x^2-1}} = \operatorname{arc} \sec x.$

21. $\int - \frac{dx}{x\sqrt{x^2-1}} = \operatorname{arc} \operatorname{cosec} x.$

22. $\int \frac{dx}{\sqrt{2x-x^2}} = \operatorname{arc} \operatorname{vers} x.$

23. $\int - \frac{dx}{\sqrt{2x-x^2}} = \operatorname{arc} \operatorname{covers} x.$

II.

RATIONAL ALGEBRAIC FORMS.

EXPRESSIONS CONTAINING $(a + bx)$.

24. $\int \frac{dx}{a+bx} = \frac{1}{b} \log (a+bx).$

25. $\int \frac{x\,dx}{a+bx} = \frac{1}{b^2} [a + bx - a \log (a+bx)].$

26. $\int \sqrt{\frac{a+x}{b+x}}\,dx = \sqrt{(a+x)(b+x)} + (a-b) \log (\sqrt{a+x} + \sqrt{b+x}).$

27. $\int \sqrt{\frac{a-x}{b+x}}\,dx = \sqrt{(a-x)(b+x)} + (a+b) \operatorname{arc} \sin \sqrt{\frac{x+b}{a+b}}.$

28. $\int \frac{x^2dx}{a+bx} = \frac{1}{b^3} [\tfrac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2 \log (a+bx)].$

29. $\int \frac{dx}{x(a+bx)} = -\frac{1}{a} \log \frac{a+bx}{x}.$

30. $\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2} \log \frac{a+bx}{x}.$

31. $\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}.$

32. $\int \frac{x\,dx}{(a+bx)^2} = \frac{1}{b^2}\left[\log(a+bx) + \frac{a}{a+bx}\right].$

33. $\int \frac{x^2dx}{(a+bx)^2} = \frac{1}{b^3}\left[a + bx - 2a\log(a+bx) - \frac{a^2}{a+bx}\right].$

34. $\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\log\left(\frac{a+bx}{x}\right).$

EXPRESSIONS CONTAINING $(a+bx^2)$.

35. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{arc\,tan}\frac{x}{a}.$

36. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}.$

37. $\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}}\operatorname{arc\,tan} x\sqrt{\frac{b}{a}},$ when $a>0$, and $b>0$.

38. $\int \frac{dx}{a^2-b^2x^2} = \frac{1}{2ab}\log\left(\frac{a+bx}{a-bx}\right),$ when $a>0$, and $b>0$.

39. $\int \frac{a\,dx}{x^2-a^2} = \log\sqrt{\frac{x-a}{x+a}}.$

40. $\int \frac{x\,dx}{a+bx^2} = \frac{1}{2b}\log\left(x^2+\frac{a}{b}\right).$

41. $\int \frac{x^2dx}{a+bx^2} = \frac{x}{b} - \sqrt{\frac{a}{b^3}}\operatorname{arc\,tan} x\sqrt{\frac{b}{a}}.$

42. $\int \frac{dx}{x^2(a+bx^2)} = -\frac{1}{ax} - \sqrt{\frac{b}{a^3}}\operatorname{arc\,tan} x\sqrt{\frac{b}{a}}.$

43. $\int \frac{dx}{(a+bx^2)^2} = \frac{x}{2a(a+bx^2)} + \frac{1}{2\sqrt{a^3b}}\operatorname{arc\,tan} x\sqrt{\frac{b}{a}}.$

44. $\int \frac{dx}{(a+bx^2)^{m+1}} = \frac{1}{2ma}\frac{x}{(a+bx^2)^m} + \frac{2m-1}{2ma}\int \frac{dx}{(a+bx^2)^m}.$

45. $\int \frac{x^2dx}{(a+bx^2)^{m+1}} = \frac{-x}{2mb(a+bx^2)^m} + \frac{1}{2mb}\int \frac{dx}{(a+bx^2)^m}.$

Expressions involving $(a + bx^n)$.

46. $\int x^m (a + bx^n)^p dx$

$$= \frac{x^{m-n+1}(a + bx^n)^{p+1} - (m - n + 1)a\int x^{m-n}(a + bx^n)^p dx}{b(np + m + 1)}.$$

47. $\int x^{-m}(a + bx^n)^p dx$

$$= \frac{x^{-m+1}(a + bx^n)^{p+1} + b(m - np - n - 1)\int x^{-m+n}(a + bx^n)^p dx}{-a(m - 1)}.$$

48. $\int x^m (a + bx^n)^p dx$

$$= \frac{x^{m+1}(a + bx^n)^p + anp\int x^m (a + bx^n)^{p-1} dx}{np + m + 1}.$$

49. $\int x^m (a + bx^n)^{-p} dx$

$$= \frac{x^{m+1}(a + bx^n)^{-p+1} - (m + n + 1 - np)\int x^m (a + bx^n)^{-p+1} dx}{an(p - 1)}.$$

III.

IRRATIONAL ALGEBRAIC FUNCTIONS.

Expressions containing $\sqrt{a + bx}$.

50. $\int \sqrt{a + bx}\, dx = \frac{2}{3b}\sqrt{(a + bx)^3}.$

51. $\int \frac{dx}{\sqrt{a + bx}} = \frac{2}{b}\sqrt{a + bx}.$

52. $\int x\sqrt{a + bx}\, dx = -\frac{2(2a - 3bx)\sqrt{(a + bx)^3}}{15b^2}.$

53. $\int \frac{x\, dx}{\sqrt{a + bx}} = -\frac{2(2a - bx)}{3b^2}\sqrt{a + bx}.$

54. $\int \frac{\sqrt{a + bx}}{x} dx = 2\sqrt{a + bx} + \sqrt{a}\log\left(\frac{\sqrt{a + bx} - \sqrt{a}}{\sqrt{a + bx} + \sqrt{a}}\right).$

55. $\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}} \log\left(\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right)$, when $a > 0$.

56. $\int \frac{x^2dx}{\sqrt{a+bx}} = \frac{2(8a^2 - 4abx + 3b^2x^2)}{15b^3}\sqrt{a+bx}.$

57. $\int \frac{dx}{x^2\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{ax} - \frac{b}{2\sqrt{a^3}} \log\left(\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right).$

EXPRESSIONS CONTAINING $\sqrt{a^2+x^2}$.

58. $\int \sqrt{a^2+x^2}\, dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log(x+\sqrt{a^2+x^2}).$

59. $\int \frac{dx}{\sqrt{a^2+x^2}} = \log(x+\sqrt{x^2+a^2}).$

60. $\int \frac{x\,dx}{\sqrt{a^2+x^2}} = \sqrt{a^2+x^2}.$

61. $\int \frac{\sqrt{a^2+x^2}}{x}\, dx = \sqrt{a^2+x^2} - a\log\left(\frac{a+\sqrt{a^2+x^2}}{x}\right).$

62. $\int \frac{dx}{x\sqrt{a^2+x^2}} = \frac{1}{a}\log\frac{x}{a+\sqrt{a^2+x^2}}.$

63. $\int \frac{x^2dx}{\sqrt{a^2+x^2}} = \frac{x}{2}\sqrt{a^2+x^2} - \frac{a^2}{2}\log(x+\sqrt{a^2+x^2}).$

64. $\int \frac{dx}{x^2\sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{a^2x}.$

65. $\int x^2\sqrt{a^2+x^2}\, dx = \frac{x}{8}(2x^2+a^2)\sqrt{x^2+a^2} - \frac{a^4}{8}\log(x+\sqrt{a^2+x^2}).$

66. $\int \frac{dx}{(a^2+x^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{a^2+x^2}}.$

67. $\int (a^2+x^2)^{\frac{3}{2}}\, dx = \frac{x}{8}(2x^2+5a^2)\sqrt{a^2+x^2} + \frac{3a^4}{8}\log(x+\sqrt{a^2+x^2}).$

68. $\int \frac{dx}{x^3\sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{2a^2x^2} + \frac{1}{2a^3}\log\left(\frac{a+\sqrt{a^2+x^2}}{x}\right).$

69. $\displaystyle\int\frac{dx}{\sqrt{(a^2+x^2)^3}}=\frac{x}{a^2\sqrt{a^2+x^2}}.$

70. $\displaystyle\int\frac{\sqrt{a^2+x^2}\,dx}{x^2}=-\frac{\sqrt{a^2+x^2}}{x}+\log x+\sqrt{a^2+x^2}.$

Expressions containing $\sqrt{a^2-x^2}$.

71. $\displaystyle\int\sqrt{a^2-x^2}\,dx=\tfrac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}\right).$

72. $\displaystyle\int\frac{dx}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{a}.$

73. $\displaystyle\int x\sqrt{a^2-x^2}\,dx=-\tfrac{1}{3}\sqrt{(a^2-x^2)^3}.$

74. $\displaystyle\int\frac{dx}{x\sqrt{a^2-x^2}}=\frac{1}{a}\log\left(\frac{x}{a+\sqrt{a^2-x^2}}\right).$

75. $\displaystyle\int\frac{\sqrt{a^2-x^2}}{x}\,dx=\sqrt{a^2-x^2}-a\log\frac{a+\sqrt{a^2-x^2}}{x}.$

76. $\displaystyle\int\frac{x\,dx}{\sqrt{a^2-x^2}}=-\sqrt{a^2-x^2}.$

77. $\displaystyle\int x^2\sqrt{a^2-x^2}\,dx=-\frac{x}{4}\sqrt{(a^2-x^2)^3}+\frac{a^2}{8}\left(x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}\right).$

78. $\displaystyle\int\frac{x^2dx}{\sqrt{a^2-x^2}}=-\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}.$

79. $\displaystyle\int\frac{dx}{x^2\sqrt{a^2-x^2}}=-\frac{\sqrt{a^2-x^2}}{a^2x}.$

80. $\displaystyle\int\frac{\sqrt{a^2-x^2}}{x^2}\,dx=-\frac{\sqrt{a^2-x^2}}{x}-\arcsin\frac{x}{a}.$

81. $\displaystyle\int\sqrt{(a^2-x^2)^3}\,dx=\tfrac{1}{4}\left[x\sqrt{(a^2-x^2)^3}+\frac{3a^2x}{2}\sqrt{a^2-x^2}+\frac{3a^4}{2}\arcsin\frac{x}{a}\right].$

82. $\displaystyle\int\frac{dx}{\sqrt{(a^2-x^2)^3}}=\frac{x}{a^2\sqrt{a^2-x^2}}.$

83. $\int \frac{x^2 dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{\sqrt{a^2-x^2}} - \text{arc} \sin \frac{x}{a}.$

84. $\int \frac{x^m dx}{\sqrt{a^2-x^2}} = -\frac{x^{m-1}}{m}(a^2-x^2)^{\frac{1}{2}} + \frac{m-1}{m} a^2 \int x^{m-2}(a^2-x^2)^{-\frac{1}{2}} dx.$

Expressions containing $\sqrt{x^2-a^2}$.

85. $\int \sqrt{x^2-a^2}\, dx = \frac{1}{2}[x\sqrt{x^2-a^2} - a^2 \log(x+\sqrt{x^2-a^2})].$

86. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log(x+\sqrt{x^2-a^2}).$

87. $\int \frac{x\, dx}{\sqrt{x^2-a^2}} = \sqrt{x^2-a^2}.$

88. $\int x\sqrt{x^2-a^2}\, dx = \frac{1}{3}\sqrt{(x^2-a^2)^3}.$

89. $\int \frac{\sqrt{x^2-a^2}}{x} dx = \sqrt{x^2-a^2} - a \,\text{arc} \cos \frac{a}{x}.$

90. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \text{arc sec} \frac{x}{a}.$

91. $\int \frac{x^2 dx}{\sqrt{x^2-a^2}} = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\log(x+\sqrt{x^2-a^2}).$

92. $\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2 x}.$

93. $\int x^2\sqrt{x^2-a^2}\, dx = \frac{x}{8}(2x^2-a^2)\sqrt{x^2-a^2} - \frac{a^4}{8}\log(x+\sqrt{x^2-a^2}).$

94. $\int \frac{dx}{x^3\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{2a^2x^2} + \frac{1}{2a^3} \text{arc sec} \frac{x}{a}.$

95. $\int \sqrt{(x^2-a^2)^3}\, dx$

$$= \frac{1}{4}\left[x\sqrt{(x^2-a^2)^3} - \frac{3a^2x}{2}\sqrt{x^2-a^2} + \frac{3a^4}{2}\log(x+\sqrt{x^2-a^2})\right].$$

96. $\displaystyle\int\frac{dx}{\sqrt{(x^2-a^2)^3}}=-\frac{x}{a^2\sqrt{x^2-a^2}}.$

97. $\displaystyle\int\frac{x^2dx}{\sqrt{(x^2-a^2)^3}}=-\frac{x}{\sqrt{x^2-a^2}}+\log\left(x+\sqrt{x^2-a^2}\right).$

98. $\displaystyle\int\frac{dx}{(x^2-a^2)^{\frac{3}{2}}}=-\frac{x}{a^2\sqrt{x^2-a^2}}.$

99. $\displaystyle\int(x^2-a^2)^{\frac{3}{2}}dx=\frac{x}{8}(2x^2-5a^2)\sqrt{x^2-a^2}+\frac{3a^4}{8}\log\left(x+\sqrt{x^2-a^2}\right).$

Expressions containing $\sqrt{2ax\pm x^2}$.

100. $\displaystyle\int\sqrt{2ax-x^2}dx=\frac{x-a}{2}\sqrt{2ax-x^2}+\frac{a^2}{2}\text{ arc ver }\frac{x}{a}.$

101. $\displaystyle\int\frac{dx}{\sqrt{2ax-x^2}}=\text{arc ver }\frac{x}{a}.$

102. $\displaystyle\int\frac{x\,dx}{\sqrt{2ax-x^2}}=-\sqrt{2ax-x^2}+a\text{ arc ver }\frac{x}{a}.$

103. $\displaystyle\int\frac{dx}{x\sqrt{2ax-x^2}}=-\frac{\sqrt{2ax-x^2}}{ax}.$

104. $\displaystyle\int x\sqrt{2ax-x^2}dx=-\frac{3a^2+ax-2x^2}{6}\sqrt{2ax-x^2}+\frac{a^3}{2}\text{ arc ver }\frac{x}{a}.$

105. $\displaystyle\int\frac{\sqrt{2ax-x^2}}{x}dx=\sqrt{2ax-x^2}+a\text{ arc ver }\frac{x}{a}.$

106. $\displaystyle\int\frac{\sqrt{2ax-x^2}}{x^3}dx=-\left(\frac{2ax-x^2}{3ax^3}\right)^{\frac{3}{2}}.$

107. $\displaystyle\int\frac{dx}{(2ax-x^2)^{\frac{3}{2}}}=\frac{x-a}{a^2\sqrt{2ax-x^2}}.$

108. $\displaystyle\int\frac{x\,dx}{(2ax-x^2)^{\frac{3}{2}}}=\frac{x}{a\sqrt{2ax-x^2}}.$

109. $\displaystyle\int\frac{dx}{\sqrt{2ax+x^2}}=\log\left(x+a+\sqrt{2ax+x^2}\right).$

110. $\int \frac{x^2 dx}{\sqrt{2ax - x^2}} = -\frac{x + 3a}{2}\sqrt{2ax - x^2} + \frac{3}{2}a^2 \text{ arc ver } \frac{x}{a}.$

111. $\int \frac{x^3 dx}{\sqrt{2ax - x^2}} = -\left(\frac{x^2}{3} + \frac{5}{6}ax + \frac{5}{2}a^2\right)\sqrt{2ax - x^2} + \frac{5}{2}a^3 \text{ arc ver } \frac{x}{a}.$

Expressions containing $\sqrt{a + bx \pm cx^2}$.

112. $\int \frac{dx}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{c}} \log (2cx + b + 2\sqrt{c}\sqrt{a + bx + cx^2}).$

113. $\int \sqrt{a + bx + cx^2}\, dx = \frac{2cx + b}{4c}\sqrt{a + bx + cx^2}$

$$- \frac{b^2 - 4ac}{8c^{\frac{3}{2}}} \log (2cx + b + 2\sqrt{c}\sqrt{a + bx + cx^2}).$$

114. $\int \frac{dx}{\sqrt{a + bx - cx^2}} = \frac{1}{\sqrt{c}} \text{ arc sin } \frac{2cx - b}{\sqrt{b^2 + 4ac}}.$

115. $\int \sqrt{a + bx - cx^2}\, dx = \frac{2cx - b}{4c}\sqrt{a + bx - cx^2}$

$$+ \frac{b^2 + 4ac}{8c^{\frac{3}{2}}} \text{ arc sin } \frac{2cx - b}{\sqrt{b^2 + 4ac}}.$$

116. $\int \frac{x\, dx}{\sqrt{a + bx + cx^2}} = \frac{\sqrt{a + bx + cx^2}}{c}$

$$- \frac{b}{2c}\left[\frac{1}{\sqrt{c}} \log (2cx + b + 2\sqrt{c}\sqrt{a + bx + cx^2})\right].$$

117. $\int \frac{x^2 dx}{\sqrt{a + bx + cx^2}} = \sqrt{a + bx + cx^2}\left(\frac{x}{2c} - \frac{3b}{4c^2}\right)$

$$+ \left(\frac{3b^2}{8c^2} - \frac{a}{2c}\right)\int \frac{dx}{\sqrt{a + bx + cx^2}}.$$

118. $\int \frac{x^n dx}{\sqrt{a + bx + cx^2}} = \frac{x^{n-1}\sqrt{a + bx + cx^2}}{nc}$

$$- \frac{n-1}{n} \cdot \frac{a}{c}\int \frac{x^{n-2} dx}{\sqrt{a + bx + cx^2}} - \frac{2n-1}{2n} \cdot \frac{b}{c}\int \frac{x^{n-1} dx}{\sqrt{a + bx + cx^2}}.$$

IV.

TRIGONOMETRIC AND TRANSCENDENTAL FUNCTIONS.

119. $\int \sin^2 x\, dx = \frac{1}{2} x - \frac{1}{4} \sin 2x.$

120. $\int \cos^2 x\, dx = \frac{1}{2} x + \frac{1}{4} \sin 2x.$

121. $\int \tan x\, dx = \log \sec x.$

122. $\int \cot x\, dx = \log \sin x.$

123. $\int \frac{dx}{\sin x} = \log \tan \frac{1}{2} x.$

124. $\int \frac{dx}{\cos x} = \log \tan \left(\frac{\pi}{4} + \frac{1}{2} x \right).$

125. $\int \operatorname{cosec} x\, dx = \log \tan \frac{1}{2} x.$

126. $\int \frac{d\theta}{a + b \cos \theta} = \frac{2}{\sqrt{a^2 - b^2}} \arctan \left[\left(\frac{a - b}{a + b} \right)^{\frac{1}{2}} \tan \frac{\theta}{2} \right]$, when $a > b$,

$$= \frac{1}{\sqrt{b^2 - a^2}} \log \frac{\sqrt{b + a} + \sqrt{b - a} \tan \frac{\theta}{2}}{\sqrt{b + a} - \sqrt{b - a} \tan \frac{\theta}{2}}, \text{ when } a < b.$$

127. $\int x \sin x\, dx = \sin x - x \cos x.$

128. $\int x^2 \sin x\, dx = 2x \sin x - (x^2 - 2) \cos x.$

129. $\int x \cos x\, dx = \cos x + x \sin x.$

130. $\int x^2 \cos x\, dx = 2x \cos x + (x^2 - 2) \sin x.$

131. $\int \frac{\sin x}{x} dx = x - \frac{x^3}{3\underline{|3}} + \frac{x^5}{5\underline{|5}} - \frac{x^7}{7\underline{|7}} + \cdots.$

132. $\int \frac{\cos x}{x} dx = \log x - \frac{x^2}{2\lfloor 2} + \frac{x^4}{4\lfloor 4} - \frac{x^6}{6\lfloor 6} + \cdots.$

133. $\int \text{arc} \sin x \, dx = x \,\text{arc} \sin x + \sqrt{1 - x^2}.$

134. $\int \text{arc} \cos x \, dx = x \,\text{arc} \cos x - \sqrt{1 - x^2}.$

135. $\int \text{arc} \tan x \, dx = x \,\text{arc} \tan x - \frac{1}{2} \log (1 + x^2).$

136. $\int \text{arc} \,\text{cotan}\, x \, dx = x \,\text{arc} \cot x + \frac{1}{2} \log (1 + x^2).$

137. $\int \text{arc} \,\text{vers}\, x \, dx = (x - 1) \,\text{arc} \,\text{vers}\, x + \sqrt{2x - x^2}.$

138. $\int \log x \, dx = x \log x - x.$

139. $\int \frac{dx}{\log x} = \log (\log x) + \log x + \frac{1}{2^2} \log^2 x + \frac{1}{2 \cdot 3^2} \log^3 x + \cdots.$

140. $\int \frac{dx}{x \log x} = \log (\log x).$

141. $\int x e^{ax} dx = \frac{e^{ax}}{a^2} (ax - 1).$

142. $\int e^{ax} \sin x \, dx = \frac{e^{ax} (a \sin x - \cos x)}{a^2 + 1}.$

143. $\int e^{ax} \cos x \, dx = \frac{e^{ax} (a \cos x + \sin x)}{a^2 + 1}.$

144. $\int e^{ax} \log x \, dx = \frac{e^{ax} \log x}{a} - \frac{1}{a} \int \frac{e^{ax}}{x} dx.$

145. $\int x^m e^{ax} dx = \frac{x^m e^{ax}}{a} - \frac{m}{a} \int x^{m-1} e^{ax} dx.$

146. $\int x^m \log x \, dx = x^{m+1} \left[\frac{\log x}{m + 1} - \frac{1}{(m + 1)^2} \right].$

147. $\int \sin^n x \, dx = - \frac{\sin^{n-1} x \cos x}{n} + \frac{n - 1}{n} \int \sin^{n-2} x \, dx.$

148. $$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx.$$

149. $$\int \cos^m x \sin^n x\,dx = \frac{\cos^{m-1}x \sin^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \cos^{m-2}x \sin^n x\,dx$$

$$= -\frac{\sin^{n-1}x \cos^{m+1}x}{m+n} + \frac{n-1}{m+n}\int \cos^m x \sin^{n-2}x\,dx.$$

150. $$\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx.$$

151. $$\int x^m \log^n x\,dx = \frac{x^{m+1}}{m+1}\log^n x - \frac{n}{m+1}\int x^m \log^{n-1}x\,dx.$$

152. $$\int \frac{x^m\,dx}{\log^n x} = -\frac{x^{m+1}}{(n-1)\log^{n-1}x} + \frac{m+1}{n-1}\int \frac{x^m\,dx}{\log^{n-1}x}.$$

153. $$\int a^{mx} x^n\,dx = \frac{a^{mx}x^n}{m\log a} - \frac{n}{m\log a}\int a^{mx}x^{n-1}\,dx.$$

154. $$\int \frac{a^x\,dx}{x^m} = -\frac{a^x}{(m-1)x^{m-1}} + \frac{\log a}{m-1}\int \frac{a^x\,dx}{x^{m-1}}.$$

155. $$\int e^{ax}\cos^n x\,dx = \frac{e^{ax}\cos^{n-1}x(a\cos x + n\sin x}{a^2+n^2}.$$

$$+\frac{n(n-1)}{a^2+n^2}\int e^{ax}\cos^{n-2}x\,dx.$$

156. $$\int x^m \cos ax\,dx = \frac{x^{m-1}}{a^2}(ax\sin ax + m\cos ax)$$

$$-\frac{m(m-1)}{a^2}\int x^{m-2}\cos ax\,dx.$$

Note A.

Assume that the surface is given as in Fig. 39, and let the point P in Fig. 45 correspond with the point P in Fig. 39.

PX', PY', and PZ' are drawn parallel to the coördinate axes. Let PS be the section of the surface made by the $X'Z'$ plane, and let PR be the section made by the $Y'Z'$ plane. Let PA' be the intersection of the plane tangent to the surface at P and the $X'Z'$ plane, and let PN' be the line cut from the same tangent plane by the $Y'Z'$ plane.

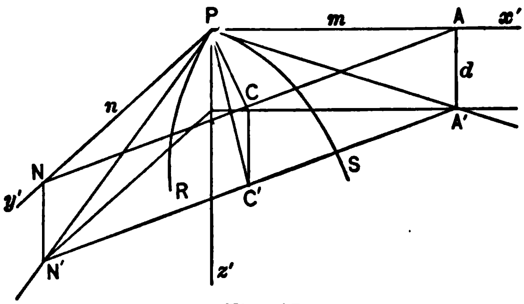

Fig. 45.

Evidently PA' and PN' determine the tangent plane at P. Now let a plane be passed parallel to the $X'Z'$ plane at a distance d below it. This plane cuts the lines PA' and PN' at A' and N' respectively. $A'A$ and $N'N$ are drawn perpendicular to PX' and PY' respectively, and the points A and N are connected. PC is drawn perpendicular to AN, then CC' is drawn perpendicular to $A'N'$, and finally the points P and C' are joined.

From the equation of the surface by Art. 42,

$$\tan APA' = \frac{\partial z}{\partial x}, \tag{1}$$

and

$$\tan NPN' = \frac{\partial z}{\partial y}. \tag{2}$$

Let $PA = m$, and $PN = n$.

Now $PC : m = n : NA = n : \sqrt{m^2 + n^2}$; $\therefore\ PC = \dfrac{mn}{\sqrt{m^2+n^2}}$.

Also
$$\tan CPC' = \frac{d}{PC} = \frac{d\sqrt{m^2+n^2}}{mn};$$

hence
$$\tan^2 CPC' = \frac{d^2}{m^2} + \frac{d^2}{n^2} = \tan^2 APA' + \tan^2 NPN'.$$

But
$$CPC' = \gamma.$$

Therefore
$$\tan^2\gamma = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2, \qquad \text{by (1) and (2)}$$

or
$$\sec^2\gamma = 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$